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Conformal Nets and Topological Modular Forms

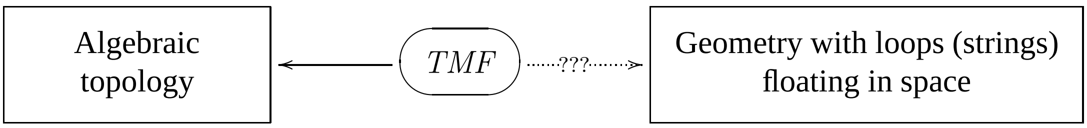
Acronym:

CN&TMF

Application for a five year (60 months) ERC starting grant

Project Summary: Some mathematical notions can be described in both algebraic and geometric ways. The geometric descriptions are sometimes richer and can reveal hidden symmetries.

K-theory is an example of a theory that has both and algebraic and geometric description. It links algebraic topology with geometry on curved spaces, to the mutual benefit of both disciplines. We believe that the notion of topological modular forms (TMF) can play a similar role by connecting algebraic topology with geometry on loop spaces. The latter is a subject with many open questions, and it is strongly influenced by string theory.



So far, TMF only has an algebraic definition; our main goal is to find a geometric definition of that theory.

The geometric definition of K-theory is done using Clifford algebras. These are a sequence of algebras, the first of which are the real numbers, the complex numbers, and the quaternions. Given the crucial role of Clifford algebras in K-theory, it is natural to look for analogs in the context of TMF. Our innovating idea is to use conformal nets in the place of Clifford algebras. Conformal nets were introduced by physicists. They form an axiomatic framework for quantum field theory, and are not so well known in the broader mathematical community.

Section 1

The principal investigator

1(a). Scientific leadership potential

Early leadership in management

While a graduate student at MIT, I organized in collaboration with Chris Douglas and Mike Hill the first *Talbot workshop*, on the Stolz-Teichner model of elliptic cohomology. The workshop took place from Monday, March 15, until Friday, March 19, 2004, in an MIT facility in Vermont, USA. It consisted of four plenary lectures given by Stephan Stolz on the history, applications, and open questions about elliptic cohomology, and twelve talks by the participants on various technical details and aspects of this subject.

The next year, John Francis joined our team. We applied for a three year grant (NSF DMS-0512714) and, expanding on our first concept, we founded the *Talbot workshop series*. The Talbot workshops provided an exceptional opportunity for the professional development of young mathematicians. These graduate students and junior faculty were able to acquire a solid view of a specific field of current research through an intensive workshop program. In total, this consisted of:

"Talbot 04": 4 day workshop on *Geometric models for elliptic cohomology*
"Talbot 05": 5 day workshop on *Geometric Langlands*
"Talbot 06": 6 day workshop on *Automorphisms of manifolds*
"Talbot 07": 6 day workshop on *Topological modular forms*

The last Talbot workshop that I organized, in 2007, was on the theory of topological modular forms (TMF). The program was carefully designed to range very quickly and pointedly from necessary background topics to presentations of open problems and areas of recent development. As such, everyone from graduate students to faculty were able to benefit from the workshop. Moreover, the program, covering a full range of material related to a single topic, bonded the participants together in an extremely productive pedagogical environment. On line proceedings of Talbot 2007 can be found at http://math.mit.edu/conferences/talbot/2007/tmfproc/.

Early leadership in research

Overall, my research interests have been rather broad. I have written articles in:

- algebraic topology [1], [2], [3], [5], [12], [13],
- representation theory [10], [11],
- algebraic geometry [6], [14],

- mathematical physics [7],
- combinatorics [4], [8],
- and group theory [9].

One of my best papers is [5]. It lies on the crossroads of algebraic topology, Lie theory, and infinite dimensional manifolds. Using methods from rational homotopy theory, I answered the following question: what does the Lie algebra / Lie group correspondence become when one replaces Lie algebras by L_∞-algebras? The answer uses a combination of infinite dimensional manifolds and simplicial sets.

The main example to which one can apply the methods of [5] is the string group, which is an infinite dimensional topological group, and at the same time a gerbe over the spin group. Gerbes are the subjects of [13], [6], and [7], each of them in a different context. In [1], I studied string manifolds from a more classical, homological point of view. Closely related to gerbes are topological stacks, which were the subject of my PhD thesis. I continued my doctoral work in [2], where I proved a structure theorem for topological stacks. The self-similar properties of fractal stacks were studied in [9], in relation with the theory of fractal groups.

My combinatorial papers [4], [8] are about cluster algebras. They emerged from my my work [11], [10] on the categorical structure of Kashiwara crystals. The above research line culminated in my paper [3], which is a homological study of the moduli spaces that appeared in [11].

1(b). Curriculum vitae

Academic positions:

2007- now	Universitair docent ($\approx$lecturer) at Utrecht.
2005-2007	SFB postdoc at Münster, in the group of Wolfgang Lück.
2003-2005	Graduate student at MIT under the supervision of Michael Hopkins.
2000-2002	Graduate student at Berkeley under the supervision of Allen Knutson.
1999-2000	Assistant teacher at the University of Geneva.
1995-1999	Undergraduate student in mathematics at the University of Geneva.

Academic degrees:

June 2005	PhD thesis at MIT under the supervision of Michael Hopkins.
June 1999	Diplôme de Mathématiques under the supervision of Vaughan Jones.

Contests and Awards:

May 2002	Charles B. Morrey Jr. award in recognition of exceptional scholarship in mathematics.

Sept. 1995 Took part in the 7th European Contest for Young Scientists in Newcastle, UK, with the work *Stellated Polyhedra and Tilings*.

May 1995 Took part in the contest "La Science Appelle les Jeunes" (national Swiss Contest for Young Scientists) with the work *Polyhèdres et pavages étoilés*. Qualified as excellent and thus selected for the European contest.

Advising experience:

I have organized multiple semester-long weekly seminars for graduate students:

- Spring 09 (with Tilman Bauer):
  Graduate student seminar on *Operads*, VU and Utrecht.
- Fall 08 (with Tilman Bauer):
  Graduate student seminar on *Stable homotopy theory*, VU and Utrecht.
- Spring 08 (with Hessel Posthuma):
  Graduate student seminar on *Twisted K-theory*, Utrecht.
- Fall 07:
  Graduate student seminar on *Infinity categories*, Utrecht.
- Spring 06 (with Michael Joachim):
  Graduate student seminar on *Loop groups and their representation theory*, Münster.
- Fall 04:
  Graduate student seminar on *Equivariant homotopy theory*, MIT.
- Spring and Fall 01 (with Nick Proudfoot):
  Graduate student colloquium *"Many cheerful facts"*, Berkeley.
- Spring 00:
  Graduate student seminar at the university of Geneva.

Besides the above mentioned activities, which were all concerned with bringing young PhD students to the level of current research, I am now involved with many supervisions in Utrecht:

- Dave Carchedi is a PhD student in Utrecht whose official adviser is of Ieke Moerdijk and who works on *Categorical aspects of topological stacks*. Professor Moerdijk and I are currently sharing his supervision.

- I am also involved in the supervision of Moerdijk's PhD student Marti Szilagyi, who works on *Higher category theory*.

- Sander Kupers just finished his twin math-physics bachelor thesis with me on *The construction of a fibered spectrum representing twisted K-theory*.

- Hans Westrik just finished his bachelor thesis with me on *The Jacobi elliptic functions*.

- Quirine Kroll is preparing her twin math-physics bachelor thesis with me on *Perfect matchings and the free fermion CFT*.

Funding ID

From 2005 to 2007, I was one of the four holders of the NSF grant DMS-0512714, for the organization of the international Talbot workshops. This three year NSF grant covered a total of USD 38250.- (12750.- per year), and was complemented by an MIT contribution of USD 9900.- (3300.- per year). Earlier this year, I have applied for a VIDI grant (of the Dutch national science foundation) on the same theme as this ERC proposal. If accepted, it will cover two PhD's and one postdoc over a period of five years.

1(c). Early achievement tracks-record

Preprints

[1] – ; Douglas, C.; Hill, M. *Homological obstructions to string orientations*, [arXiv:0810.2131], submitted to Geometry and Topology.
[2] – ; Gepner, D. *Homotopy Theory of Orbispaces*, [arXiv:AT/0701916].

Papers in international (refereed) journals

[3] – ; Etingof, P.; Kamnitzer, J.; Rains, E. *The cohomology ring of the real locus of the moduli space of stable curves of genus 0 with marked points*, accepted in Annals of Math.
[4] – ; Speyer, D. *The Multidimensional Cube Recurrence*, [arXiv:0708.2478], accepted in Advances in Mathematics.
[5] – *Integrating L_∞-algebras*, Compositio Mathematica, 144, (2008), no. 4, 1017–1045
[6] – ; Felder, G.; Rossi, C; Zhu, C. *A Gerbe for the Elliptic Gamma Function*, Duke Mathematical Journal 141 (2008) no.1, 1–74.
[7] – ; Ando, M.; Hellerman S., Pantev, T.; Sharpe, E. *Cluster Decomposition, T-duality, and Gerby CFT's*, Advances in Theoretical and Mathematical Physics 11 (2007), no. 5.
[8] – *A periodicity theorem for the octahedron recurrence*, J. Algebraic Combin. 26 (2007), no. 1, 1–26.
[9] – ; Bartholdi, L.; Nekrashevych, V. *Automata, Groups, Limit Spaces, and Tilings*, Journal of Algebra 305 (2006), no. 2, 629–663.
[10] – ; Kamnitzer, J. *The octahedron recurrence and $\mathfrak{gl}_n$-crystals*, Advances in Mathematics 206 (2006), no. 1, 211–249.
[11] – ; Kamnitzer, J. *Crystals and coboundary categories*, Duke Mathematical Journal 132 (2006), no. 2, 191–216.
[12] – ; Harada, M.; Holm, T. *Computation of generalized equivariant cohomologies of Kac-Moody flag varieties*, Advances in Mathematics 197 (2005), no. 1, 198–221.
[13] – ; Metzler, D. *Presentations of noneffective orbifolds*, Transactions of the A.M.S. vol 356, number 6, (2004) 2481-2499.
[14] – *An analogue of convexity for complements of amoebas of varieties of higher codimension, an answer to a question asked by B. Sturmfels*, Adv. in Geom. 4 (2004), 61-73.

Total number of coauthors: 17.

Note: The journal *Annals of mathematics* is one of the five best across all areas of mathematics. The journals *Advances in Mathematics*, *Compositio Mathematica*, and *Duke Mathematical Journal* are among the top fifteen. Publication [7] is of interdisciplinary nature, written with two physicists as coauthors.

Invited presentations at international conferences

June 09: *Invertible conformal nets,* Strings, Fields, and Topology, Oberwolfach.
May 09: *An Elmendorf's theorem for orbifolds*. Manifold perspectives, Oberwolfach.
Nov. 08: *3-categories for the working mathematician*. Higher structures, Lausanne.
Sept. 08: *From Elliptic cohomology to conformal nets*. Operator Alg. and CFT, Vienna.
Aug. 08: *TMF and the String group*. Moment maps, Lausanne.
April 08: *A uniform construction of $Spin$ and $String$*. NRW Topology Meeting, Münster.
Jan. 08: *Conformal nets form a 3-category*. Constructive Aspects of QFT, Göttingen.
Oct. 07: *Connections on string bundles*. Lie algebroids and Lie groupoids, Bakewell.
June 07: *Higher Clifford algebras*. Algebraic topology old and new, Bedlewo.
May 07: *An action of $\pi_1(\bar{M}_{0,n}(\mathbb{R}))$*. Poisson geometry and Applications, Oberwolfach.
Oct. 06: *The Dirac construction: a map from $Rep^\tau(LG)$ to $K^\tau_G(G)$*. Arbeitsgemeinschaft on Twisted K-Theory, Oberwolfach.
Sept. 06: *Orbivariant K-theory*. Topology, Oberwolfach.
Mar. 06: *An operad coming from representation theory*. Alpine Operad Workshop.
Jan. 06: *Crystals and coboundary categories*. K-Theory and Operads, Strasbourg.
June 05: *Vector bundles on orbispaces*. Pure and Applied Topology, Isle of Skye.
June 05: *A model for the String group*. Geometric Topology and QFT, Oberwolfach.
Sept. 04: *T-equivariant complex oriented cohomology of generalized flag varieties*. Elliptic cohomology and loop spaces, Fields Institute.
June 04: *T-equivariant cohomology of even dimensional cell complexes*. Geneva.

Invited talks

Mar. 09: *Towards a geometric description of TMF*. Harvard.
Geometric string structures. Harvard.
Mar. 09: *A 3-category of conformal nets*. MIT.
Jan. 09: *The free fermions: a coordinate free description*. Berkeley.
Dec. 08: *A Lie algebra for the String group*. Louvain-la-Neuve.
Aug. 08: *The String group*. Colloquium talk at ETH, Zürich.
Feb. 08: *Integrable recurrences*. Imperial Colledge.
July 07: *Les CFT forment une 3-catégorie*. Geneva.
May 07: *Orbifolds and the Borel construction*. Colloquium talk at Regensburg.
April 07: *Higher Clifford algebras*. Austin.
Jan. 07: *La périodicité de la récurrence de l'octahèdre*. Paris VII.
Dec. 06: *Higher Clifford algebras*. EPFL, Lausanne.
Nov. 06: *The homotopy theory of orbispaces*. Stanford.

Nov. 06: *Higher Clifford algebras*. Stanford.
June 06: *La périodicité de la récurrence de l'octahèdre*. Geneva.
Feb. 06: *An Adams spectral seq. for computing tmf-homology at the prime 3.* Berkeley.
Feb. 06: *Two models for the String group: a differential geometric and an algebraic geometric.* Berkeley.
Jan. 06: *Crystals and coboundary categories*. Strasbourg.
Dec. 05: *The String group and its Lie algebra*. Bonn.
Nov. 05: *A overgroup of the mapping class group*. EPFL, Lausanne.
Feb. 05: *Orbifolds are global quotients*. University of Chicago.
Feb. 05: *The String group and its Lie algebra*. Northwestern University.

Organization of international conferences

In the years 2004-07, while we were still graduate students, C. Douglas, J. Francis, M. Hill and I founded and organized the *Talbot workshop series*, whose primary purpose is to introduce young mathematicians to a field of active research. The workshops 2005-07 were supported by NSF grant DMS-0512714.

The Talbot workshop series has gained international renown, and continues to run until now. More information can be found at http://math.mit.edu/conferences/talbot/.

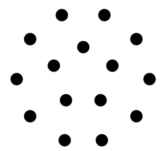
Extended Synopsis of the scientific proposal

Introduction

Mathematical notions can often be described in more than one way. Some are more algebraic, and some are more geometric. We illustrate this by an easy example: the number *fifteen* has an algebraic description given by concatenating "1" and "5". A geometric counterpart is given by exhibiting a collection with fifteen elements.

" 15 "

algebraic definition



geometric definition

The geometric description might reveal hidden symmetries, and is thus often richer.

K-theory belongs to the branch of mathematics called algebraic topology. The latter is concerned with those aspects of geometric shapes that can be studied by purely algebraic means. Thus, on it own, algebraic topology might neglect useful geometric information. But K-theory also has a geometric description. It therefore constitutes a bridge between algebraic topology and geometry, even analysis, to the mutual benefit of all subjects.

$$\begin{tikzcd} \boxed{\text{Algebraic topology}} & K\text{-theory} \arrow[l] \arrow[r] & \boxed{\text{Geometry \& Analysis on curved spaces}} \end{tikzcd}$$

This connection is only possible because K-theory has two descriptions: one algebraic and one geometric.

We believe that the standard notion of topological modular forms (TMF) can play a similar role by connecting algebraic topology with analysis on loop spaces.

$$\begin{tikzcd} \boxed{\text{Algebraic topology}} & TMF \arrow[l] \arrow[r, dotted, "???"] & \boxed{\text{Analysis with loops (strings) floating in space}} \end{tikzcd}$$

But so far, TMF only has an algebraic description. For the above purpose, one would also need a geometric one. That is the main goal of this research project:

Our main goal: Find the first geometric definition of TMF

The geometric definition of K-theory is done using Clifford algebras: these are a sequence of algebras, the first of which are the real numbers, the complex numbers, and the quaternions:

$$\text{Clifford algebras:} \qquad \mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}, \quad \ldots$$

Given the crucial role of Clifford algebras in K-theory, it is natural to look for analogs in the context of TMF. Our innovating idea is to use conformal nets in the place of Clifford algebras. Conformal nets were first introduced by physicists. They form an axiomatic framework for quantum field theory, and are not so well known in the broader mathematical community.

State of the art: K-theory and TMF

K-theory is an old and well developed subject of mathematics. Much of its glory came from the famous Atiyah-Singer index theorem, proved in 1963. That theorem is a fundamental and extremely widely used result which brings the computational strength of topology into the of world analysis. Namely, it can predict the number of dimensions of the space of solutions of certain differential equations. Its authors were jointly accorded the Abel prize in 2004 “for their discovery and proof of the index theorem, bringing together topology, geometry and analysis, and their outstanding role in building new bridges between mathematics and theoretical physics”.

TMF bears strong analogies with K-theory (real K-theory, to be precise) and has been also referred to as “higher real K-theory”. On the other hand, it is known since the work of Witten [Wi] that TMF should have something to do with loop spaces. Based on those pieces of evidence, the following heuristic definition of TMF was proposed:

$$\text{“} \quad TMF^*(M) := K^*_{\mathit{Diff}(S^1)}(LM) \quad \text{”}$$

Unfortunately, the $\mathit{Diff}(S^1)$-equivariant K-theory of the loop space is not a well defined notion. Nevertheless, the above formula remains very inspirational, and hints at the possible geometric nature of TMF.

Historically, the theory TMF grew out of the notion of elliptic cohomology, a subject that dates back from the eighties. TMF is a cohomology theory that contains in it the information of all elliptic cohomology theories put together. The existence of such a universal theory was announced in the nineties, but its construction was so intricate that the foundational papers [HMi], [HMa] never got finished. Nevertheless, the results of Hopkins, Mahowald, and Miller attracted a lot of attention, and numerous astounding results were announced [AHS], [Ho], [Be]. Lurie’s novel approach to TMF [Lu1] also allowed for remarkable applications [BeL].

The current definition of TMF is of purely algebraic nature. But given the analogy with K-theory, it is widely expected that there should also exist geometric descriptions of TMF. Indeed, many people have tried to find a geometric definition of that theory: [BDR], [HK], [ST1], [ST2]. But so far, even though a lot of progress has been made, none of those approaches were completely successful. In the next section, we will present the new idea which lets us believe that this goal is now within reach.

Given all the attempts, there’s no doubt that our project is ambitious: it would be a ground-breaking achievement with important consequences for the theory of loop spaces.

A new idea from quantum field theory: Conformal nets

Nets of von Neumann algebras were initially developed for the purpose of quantum field theory in four dimensions [Ha]. But despite their inventors expectations, they turned up most useful in the context of 2-dimensional physics, more specifically conformal field theory. These are the so-called conformal nets.

Our interest in conformal nets originated from the following question, that we heard from Stefan Stolz and Peter Teichner:

Question. *The Clifford algebras $\mathit{Cliff}(n)$ are very important in setting up K-theory. Which mathematical objects are their analogs in the context of TMF?*

We claim to have an answer to that question: the free fermion conformal net $\mathit{Fer}(n)$. Physically speaking, the free fermion is a quantum field theory that describes n massless particles with no interactions, and the associated conformal net is among the simplest ones. A physicist would certainly agree that $\mathit{Fer}(n)$ is a good analog of $\mathit{Cliff}(n)$: the former is second quantization of the latter:

$$\boxed{\mathit{Cliff}(n)} \xrightarrow{\text{second quantization}} \boxed{\mathit{Fer}(n)}$$

But we also have gathered mathematical evidence which lets us believe that $\mathit{Fer}(n)$ is the correct answer. We display it in the following table:

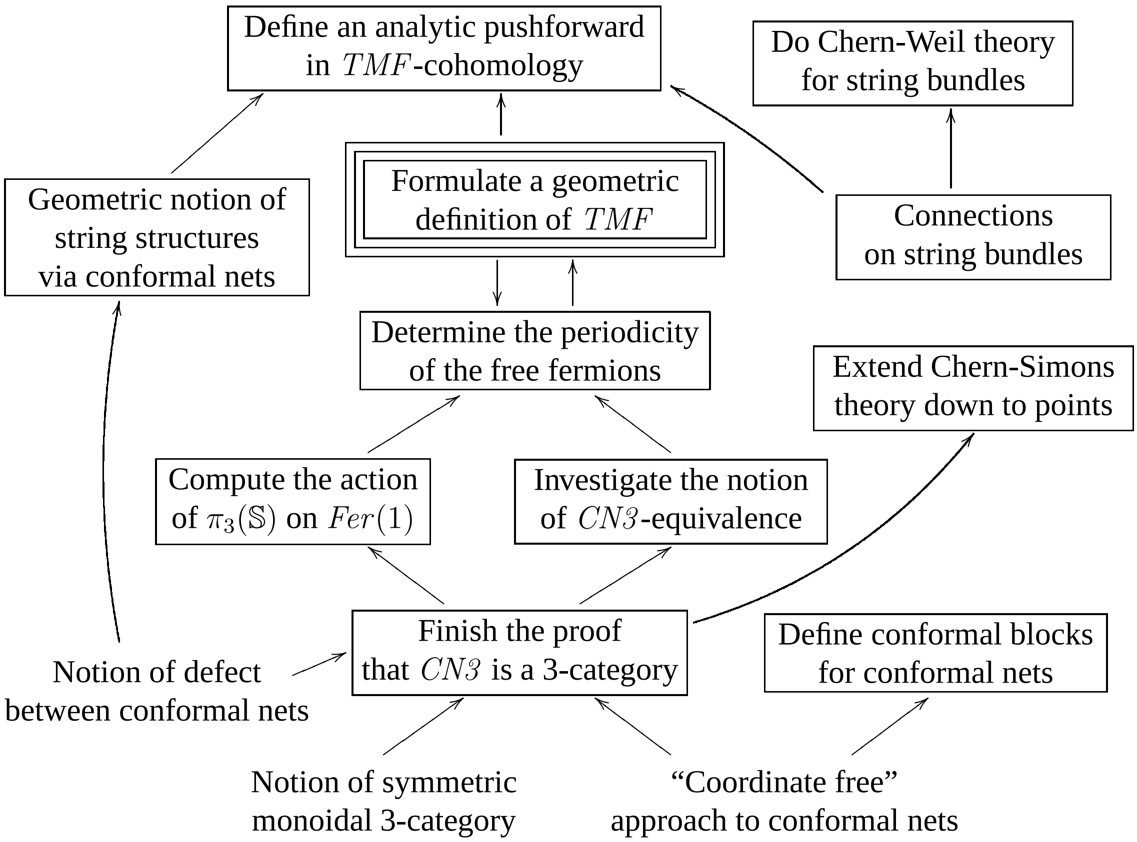
EVIDENCE THAT THE CONFORMAL NET $\mathit{Fer}(n)$ IS A GOOD ANALOG OF $\mathit{Cliff}(n)$	
Clifford algebra $\mathit{Cliff}(n)$	The free fermion $\mathit{Fer}(n)$
$\mathit{Cliff}(n)$ has an action of $O(n)$	$\mathit{Fer}(n)$ has an action of $O(n)$
Cliff is a multiplicative functor: $\mathit{Cliff}(V \oplus W) = \mathit{Cliff}(V) \otimes \mathit{Cliff}(W)$	Fer is a multiplicative functor: $\mathit{Fer}(V \oplus W) = \mathit{Fer}(V) \otimes \mathit{Fer}(W)$
Algebras form a 2-category	Conformal nets form a 3-category
$\mathit{Cliff}(n)$ can be used to define $\mathit{Spin}(n)$	$\mathit{Fer}(n)$ can be used to define $\mathit{String}(n)$

The above observations provide strong evidence for the connection between TMF and theoretical physics. That point of view was already present since the work of Witten [Wi], where he computed the index of a hypothetical Dirac operator on loop spaces and found that it ought to be a modular form. That index was never constructed mathematically, but Ando-Hopkins-Rezk [AHR] were able to give an alternative definition using solely algebraic topology. They identified it with an element of $\pi_*(\mathit{TMF})$, which then also provided extra information, not known to the physicists.

The 3-category formed by conformal nets is a structure that is quite remarkable in itself. As a mall piece of that, we have a new notion of defects between conformal nets, a concept that has already proved very useful in conformal field theory [FRS], [LR]. Given all this rich structure, we also want to study conformal nets for their own sake.

Key objectives

In the following big diagram, we have organized our various intermediate goals, along with their interrelationships. The items listed on the bottom (notion of defect, symmetric monoidal 3-category, coordinate free nets) have already been established by us, and constitute solid ground for further investigations. The boxed items represent our different objectives, listed hierarchically, and the central box represents our main goal.



We explain some of the above:

▸ Finish the proof that $CN3$ is a 3-category: We call the 3-category of conformal nets $CN3$. Our project with Bartels and Douglas is to show that $CN3$ is indeed a 3-category.

▸ Compute the action of $\pi_3(\mathbb{S})$ on $Fer(1)$: The stable homotopy groups of spheres act on the invertible objects of any 3-category. What does this action look like on $Fer(1)$?

▸ Investigate the notion of $CN3$-equivalence: The 3-categorical structure of conformal nets induces a new notion of equivalence on them. This has not been studied till now.

▸ Determine the periodicity of the free fermions: The 8-fold periodicities of K-theory and of $Cliff(n)$ are closely related. What does TMF's 576-periodicity mean for $Fer(n)$?

▸ Geometric string structures; connections on string bundles: String structures can be defined in many ways and are crucial for TMF. Not all make it easy to define connections.

▸ Define an analytic pushforward in TMF-cohomology: This is by far our most ambitious goal. It would be an TMF-analog of the Atiyah-Singer index theorem.

Resources

For this project, I plan to hire one postdoc (well acquainted with conformal nets) and three PhD students. The postdoc can help investigate the notion of $CN3$-equivalence. PhD problems include: Chern-Weil theory for string bundles, conformal blocks for conformal nets, and extendend Chern-Simons theory. The other direct costs include traveling, inviting international guests, books, and computers. Here is a summary of the budget:

PI (full time over 5 years)	370866
Post doc (3 years)	178080
PhD students (3 $\times$ 4 years)	502422
Other direct costs	99500
Indirect costs	230174
Total:	**1381042**

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[ST2] Stolz, S.; Teichner, P. *Super symmetric Euclidean field theories and generalized cohomology, a survey*, preprint, 2008.
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