VISION

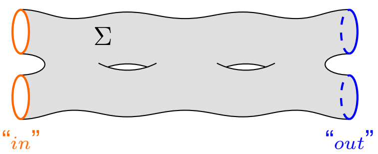
Main scientific vision The main objective of this project is to provide the first general method for producing chiral conformal Field Theories (CFTs) in the functorial formalism pioneered by Graeme Segal. The new CFTs constructed will include the WZW models and the minimal models, resolving a major challenge going right back to Segal's early foundational work. Additionally, we will strengthen the connections between vertex operator algebras (VOAs), functorial CFTs, and conformal nets.

Our main new tool for constructing functorial CFTs is the extended complex cobordism category, which includes cobordisms with thin parts, and cobordisms with cusps (illustrated in §B3).

Background: the work of G. Segal. Two-dimensional Conformal Field Theory emanated from the 1984 seminal work of the three physicists Belavin, Polyakov and Zamolodchikov [BPZ84]. These are scale invariant theories, and have in 2d an infinite dimensional symmetry given by the Virasoro algebra which is understood well enough to perform many physically meaningful computations. However, finding a suitable mathematical formalism for its rigorous development is a major challenge.

Three years later, Segal wrote a highly influential preprint [Seg87] in which he formulated a simple and clarifying idea: a CFT is a functor from a category of conformal bordisms to the category of Hilbert spaces and bounded linear maps:

$$\{\ \text{Conformal bordisms}\ \} \longrightarrow \{\ \text{Hilbert spaces}\ \}$$



$$\mapsto \qquad Z_\Sigma : H_{in} \to H_{out} \tag{1}$$

But Segal's preprint, even though it was published, remains unfinished to this day and neither Segal, nor anyone else, could construct[1] some of the main examples that everyone believed should exist (e.g. the WZW models): describing the linear operators (1) and proving that they are bounded turned out to be notoriously difficult.

In the meantime, other mathematical descriptions of CFT had appeared in the literature. In 1986, motivated by the work of Frenkel, Lepowsky, Meurman on the monster sporadic group [FLM88], Borcherds introduced the notion of a vertex operator algebra (VOA) [Bor86]. And some years later, Gabbiani, Frölich and Wasserman introduced conformal nets [GaFro93, Was98] as an approach to CFT in the spirit of Haag and Kastler's operator algebraic approach to quantum field theory [Haa92, HaKas64]. It is expected that VOAs, conformal nets, and functorial CFTs should be more-or-less equivalent. There should therefore exist of multitude of chiral CFTs.

While vertex operator algebras and conformal nets have been flourishing areas of research for many decades, examples of CFTs fitting in Segal's functorial formalism remain vanishingly rare. Even the 'trivial' examples of free boson [Pos12] and free fermion CFTs [Ten17] required a substantial amount to work to bring within the scope of Segal's formalism. The only other examples are the remarkable recent constructions of Liouville CFT and its non-unitary counterpart [GKRV21, GKR23].

Among the most important and basic CFT that are still missing are the so-called WZW

[1] Here, *constructing* a CFT includes proving the Segal axioms (a.k.a. conformal bootstrap).

models, and the minimal models. Indeed, the WZW models are the topic of the famously still unwritten Chapter 11 of [Seg87]. Successfully realising the vision of this project – constructing these crucial WZW models and many many more – will complete Segal's project, and finally give the range of examples needed to validate this rigorous approach to conformal theory.

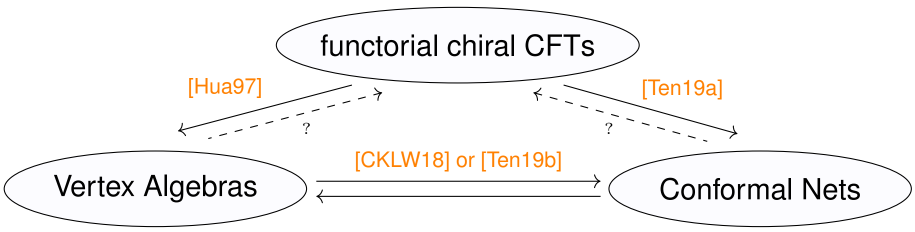
Chiral CFTs, VOAs, and conformal nets. This project concerns exclusively chiral CFTs, and by a "conformal net" we shall always mean a chiral conformal net, as in [GaFro93] or [CKLW18].

TABLE 1: DEFINITION SUMMARY OF FUNCTORIAL CHIRAL CFT

a linear category $\mathcal{C}(S)$ for every circle S.	a forgetful functor $\mathcal{C}(S) \to$ Hilb.
a functor $F_\Sigma : \mathcal{C}(S_{in}) \to \mathcal{C}(S_{out})$ for every cobordism $\Sigma : S_{in} \to S_{out}$	a bounded linear map $H \to F_\Sigma(H)$ for every object $H \in \mathcal{C}(S_{in})$
trivialisations of F_Σ when Σ is an annulus.	a certain holomorphicity condition

It is worth noting that while Segal's seminal paper mentions both full and chiral CFTs, some important aspects of the definition of chiral CFT were missing from [Seg87] (specifically: the bottom row of Table 1). A complete definition appeared for the first time in my lecture notes [Hen20]. It is reproduced in summary form in Table 1 above.

Returning to the expected equivalence between VOAs, conformal nets, and functorial CFTs, we summarise the current state of the art in the diagram below:



The downward arrows (by Huang and Tener, involving variations on the axioms), while technical, should be thought of as 'forgetful functors' as Segal's formalism includes quite a lot more data. Indeed, this is why functorial CFTs are so difficult to construct. We expect this extra data can be reconstructed from the VOA or conformal net. There are also important works [CKLW18], [Ten19b] addressing the horizontal arrows, which currently only work under a technical condition called strong locality. Our main goals are:

- Build the dashed upward-pointing arrows in the above triangle diagram,
- Upgrade the horizontal arrows relating vertex algebras and conformal nets by showing that all conformal nets have associated VOAs, and better characterizing those VOAs that come from conformal nets.

To summarise, while there are now ways of constructing VOAs and conformal nets from functorial CFTs, functorial CFTs have proven much more difficult to construct. Our main goal is to provide the missing constructions, and to reinforce the existing ones:

Main scientific objective: Construct functorial chiral CFTs from chiral conformal nets and VOAs, and prove the equivalence of the 3 pictures of chiral conformal field theory.

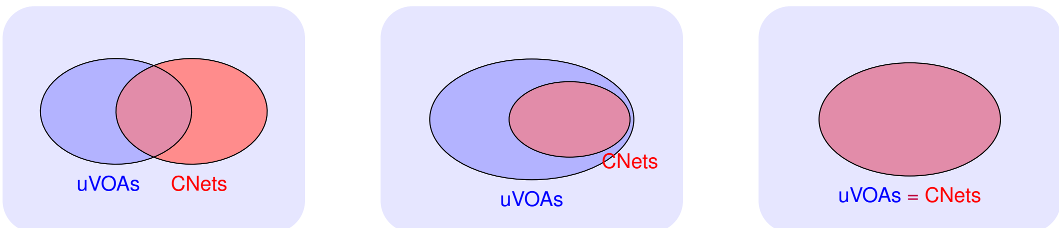
National Importance / Impact. Bringing together the UK's strengths in mathematical physics, operator algebras and topology, this project will cement the UK's leading position in the rigorous formulation of conformal field theory, establishing Oxford as a world leading centre on this interface. The project will strengthen the pipeline of UK trained future talent, both through supporting the PDRA in their development into a leading scientist, and additionally through the impact it will inevitably have on the community of around 10 postdoctoral researchers and graduate students in Oxford with adjacent interests. The project ideally complements other UK currently funded research in field theories (Dimofte, Schafer-Nameki). The rigorous approach taken here contrasts with funded research in physics (Russo) and connects to the recent project of Evans on the interface between conformal field theories and operator algebras.

In addition to the benefits to the Oxford and the UK research communities, the primary short-term beneficiaries of this project are other researchers worldwide in the CFT and more broad mathematical physics community (e.g. the groups in Germany, Italy, and United States) as this project will bring three very different strands of research together. In the longer term, the impact of pure research of this nature, while hard to predict, hinges on the rapid and wide dissemination of results amongst the research community. Indeed, there is a long track record of research in mathematical physics impacting significantly on society at large.

APPROACH

The precise scientific objectives of this proposal are divided into three strands: **A**, **B**, and **C**. We later subdivide them into sub-strands **A1–A3**, **B1–B4**, and **C1–C4**, each describing sub-objectives, along with the concrete steps needed to achieve them.

<u>Strand A:</u> Strengthening the connection between VOAs and conformal nets. We will provide fully general construction of a unitary VOA from a conformal net and a conceptual re-characterisation of strong locality which will unify the existing constructions of conformal nets from unitary VOAs. We hope these new tools will be used to resolve the question of whether strong locality is automatic.



before *by my proposed work* *conjecturally*

<u>Strand B:</u> Constructing CFTs from conformal nets. This is the core of the proposal. We will construct CFTs associated to the (super-)conformal nets in a certain wide class which contains the free fermion conformal net, and which is closed under various natural operations. This class includes:

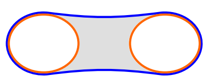
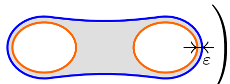
- All chiral WZW conformal nets.
- All lattice conformal nets
- All unitary minimal models with $c < 1$.
- Unitary $\mathcal{N}=2$ minimal models with $c < 3$.
- All orbifolds of the above models.
- All susy chiral WZW conformal nets.
- The Moonshine net.
- Unitary $\mathcal{N}=1$ minimal models with $c < 2$.
- All cosets of the above models.
- All sub-nets of the above.

This will represent a vast advance, compared to the few presently known constructions.

<u>**Strand C:**</u> **Stress testing the definitions and constructions.** It is vital that the various constructions between the 3 pictures of unitary CFTs be compatible with each other. This strand will provide researchers the essential tools enabling them to move between conformal nets, unitary VOAs, and unitary functorial CFTs. This will be done by formalising the notion of unitary chiral CFT à la Segal, reworking Huang's construction of a VOA from a CFT in the unitary setting, and connecting functorial CFTs with modular functors.

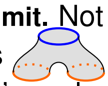
▷ **A. Strengthening the connection between VOAs and conformal nets.**
A1. A fully general construction of a VOA from a conformal net. We will improve on [CKLW18] by showing that <u>every</u> conformal net has an associated VOA. Note that many of the existing results and constructions on conformal nets require technical assumptions, such as rationality, or strong additivity. This will be in stark contrast with what we plan to do, which will work with no assumptions at all. The underlying vector space of the VOA will be given by the finite-energy vectors of the vacuum sector of a conformal net, with the Virasoro vector given by $L_{-2}\Omega$. A crucial technical step toward that goal will be to prove that every finite energy vector is a sum of two algebra elements coming from the conformal net. If the conformal net is in the image of the [CKLW18] map, we will show that our construction recovers the one in loc. cit.
A2. From VOAs to conformal nets: show that the existing construction agree. There currently exist two very different ways of constructing a conformal net starting from a unitary VOA. The original construction, by [CKLW18], involves non-linear tools (functional calculus), and requires a technical assumption called strong locality. The more recent approach, by Tener [Ten19b, Ten19c], is purely linear. The underlying technical assumption is of a very different nature: bounded localised vertex operators. The latter approach is much more concrete, as it gives explicit elements in the local von Neumann algebras of the conformal net. We will show that Tener's technical assumption is equivalent to strong locality, and that the two constructions are local with respect to each other. This will force them to agree.
A3. Determine whether strong locality is automatic. The work of A1 and A2 will shed new insight on strong locality. This will allow us to tackle the major challenge of whether this condition is automatic for all unitary VOAs (we suspect strong locality is automatic).

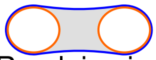
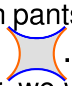
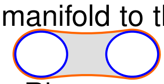
▷ **B. Constructing CFTs from conformal nets.** The functorial CFTs that we will construct will all satisfy a specially nice property of 'being bounded in the thin limit'. That property states that the evolution operator associated to a 'pair of pants'  = , remains a bounded operator in the limit as the pair of pants becomes thinner. We can then set:

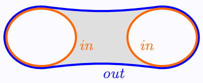
$Z\Big($  $\Big) = \lim_{\varepsilon\to 0} Z\Big($  $\Big).$

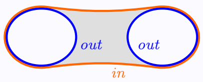
B1. Boundedness in the thin limit for the free fermion CFT. As a first step, to get the machine started, we will establish that the CFT constructed by Tener in [Ten17] satisfies this key analytic property of being bounded in the thin limit. This will allow us to more firmly connect the functorial CFT constructed by Tener to the free fermion conformal net.

B2. Study conformal nets which are bounded in the thin limit. We will identify the analogous condition to being bounded in the thin limit in the language of conformal nets, and we will establish inheritance properties of the class of conformal nets which are bounded in the thin limit. Specifically, we will show that this property is inherited by (a) tensor products, (b) passing to subnets, and (c) simple current extensions. Together with the fact that the free fermion conformal net is bounded in the thin limit, this will establish this property for a very large class of conformal nets: all those listed in the above bullet list.

B3. Construct a CFT from a conformal net which is bounded in the thin limit. Not every surface can be constructed by gluing together copies of the pair of pants , because the number of blue outgoing circles will always remain one (recall we're only allowed to glue blue circles to red circles), and the genus will never increase.

We will show that the evolution operator associated to thin pants  is equivalent to the data of some other linear operator, associated to . Re-gluing in the other way a 1-dimensional manifold to the above four-pointed star, we will construct linear operators associated to . From there, assembling thin pants and copants, one can construct arbitrary Riemann surfaces:

Gluing copies of 

& 

$\Rightarrow$ all higher genus surfaces.

B4. Establish the Segal commutation relations for the CFTs constructed in B3, and relate these to the VOA from A1. There is a direct way of going from VOAs to functorial CFTs, by declaring the CFT evolution operators to be the universal solutions of the so-called 'Segal Commutation relations'. But, as noted in of [Tel95] (p.274), establishing the desired formal properties of this construction –compatibility with gluing of cobordisms– has been a major open question. We'll show that our construction of functorial CFT produces universal solutions to the Segal Commutation relations, thus solving this major open question.

▷ C. Stress testing the definitions and constructions.

C1. Formalise unititary for chiral CFTs. Chiral CFT were introduced in Segal's original paper [Seg87], but were not fully formally defined. The latter was done for the first time by me in [Hen20]. The unitary analogs of these definitions is subtle in unexpected ways, and a goal of this project will be to shed light on those subtleties. One such unexpected subtlety is the need to work with Hilbert spaces whose norm is only well defined up a global positive-scalar ambiguity.

C2. Rework Huang's construction of a VOA from a CFT in the unitary setting. In his celebrated thesis [Hua97], Huang provided a geometric way of thinking about VOAs by explaining how they arise starting from a genus-zero CFT. If the CFT is unitary, it is reasonable to expect that the corresponding VOA should be unitary too. The goal of his project would be to show that this is indeed the case. This project, while not offering too many challenges, would be an excellent test of the notion of unitary chiral CFT.

C3. The modular functor associated to a CFT. An important consistency check for the definition of chiral CFT is for it to admit a forgetful functor down to the notion of modular functor defined in [BK01]. Specifically, the left-most column of Table 1 (on p.2 of this proposal) should be equivalent to the notion of modular functor. We expect this project to be approachable with the same tools used to prove the equivalence between topological modular functors and complex modular functors in loc. cit.

C4. Agreement of the constructions (C2) ∘ (B3) = (A1). Starting from a conformal net which is bounded in the thin limit, applying the construction in (B3), one obtains a chiral CFT to which one can apply the unitary analog of Huang's construction. It will be important to check that this agrees with the construction of a VOA from a conformal nets outlined in (A1). To do so, we will formulate and establish a recognition principle, allowing us to recognise the VOA associated to a conformal net.

Feasibility and risk management. The three main strands **A1–A3**, **B1–B4**, and **C1–C4** are designed to be broadly independent of each other, so that obstruction or delay in one strand does not impeed overall progress. This is illustrated in the diagrammatic workflow on next page.

Dissemination. The research outputs of the project will be posted on the arXiv, ensuring their wide visibility. Each of the three main strands of this project will produce at least two high quality papers, which can be submitted to leading journals. The planned conference, towards the end of the project, will provide further dissemination opportunities bringing together leading experts in conformal field theory to maximise the long term benefits of the outcomes of this project.

Research environment. This project lies at the crossroads of mathematical physics, topology, and operator algebras, and Oxford's Mathematical Institute offers the ideal research environment to deliver it. Both the mathematical physics group, and the topology groups are extremely large (mathematical physics currently has three ERCs) and they comprise an impressive number of world leading scientists, and highly motivated post-docs. On the operator algebra side, S. White, currently holder of an ERC, M. Gubinelli, and their postdocs, will offer invaluable expertise.

In addition to the large number of external seminars run by these groups, there are many informal interactions. Of particular relevance to this project is the weekly meeting bringing together members of the Mathematical Institute around myself and S. Schafer-Nameki, and members of the physics department around S. Simon, and P. Fendley.

Financial management. The financial aspects of the proposal will be overseen by the Mathematical Institute's Finance Officer, in conjunction with the University's Research Services Office.

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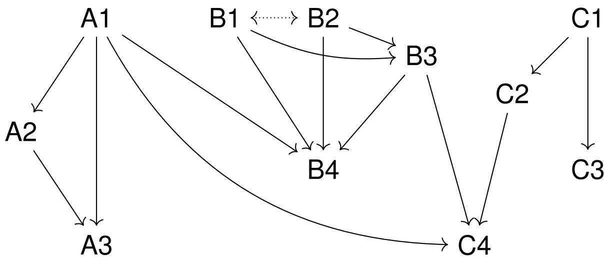
DIAGRAMMATIC WORKPLAN

The following flow-chart describes the various interdependence of the sub-objectives that we have formulated. A solid arrow indicates that the methods or arguments in the first topic are likely to be useful in the second topic. Dotted arrows indicate problems that do not formally depend on each other, but that are best tackled simultaneously, as they only make sense in the presence of the other:



The diagram below provides an approximate timeline of the various parts of the project over the 3.5 year duration of the grant:

The PDRA will be hired shortly after the beginning of the project, and will be instrumental throughout the length of the project. They will bring technical expertise from either VOA theory, or conformal net theory, and will get familiar with the other approaches to CFT throughout the length of they stay.

J. Tener will be actively involved in this project throughout its length. He will visit Oxford for a period of six months, after some of the initial sub-projects have been completed.

Finally, I will organise a twinned conference on CFT towards the end, to maximise the impact of the project. This will bring together mathematicians working in the three frameworks of CFT, along with mathematically oriented physicists. The conference will be similar in spirit and format to the workshop https://www.conferences.uni-hamburg.de/event/421/overview which I organised in March 2024, in Hamburg and in Toronto.