Rational generalised 2-equivariant elliptic cohomology

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October 2016

1 Generalised equivariant elliptic cohomology

Elliptic cohomology is a familiy of cohomology theories. There is one elliptic cohomology for every elliptic curve E, subject to a certain condition called Landweber exactness. When working over rings of characteristic zero the Landweber exactness condition is trivially satisfied, and every elliptic curve E (over any $\mathbb{Q}$-algebra k) has an associated elliptic cohomology theory. We denote it $\mathcal{E}\ell\ell_E^*$. Its ring of coefficients is given by

$$\mathcal{E}\ell\ell_E^*(pt) = k[\omega^{\pm 1}] := \bigoplus_{n\in\mathbb{Z}} \omega^{\otimes n},$$

where ω is the k-module of invariant differentials on E.

Given a bicommutant category $\mathcal{T}$, we have reasons to believe that there exists such a thing as *$\mathcal{T}$-equivariant elliptic cohomology.* In this article, we construct such a theory when the ring of definition of the elliptic curve has characteristic zero and when the bicommutant category satisfies a certain finiteness condition. The corresponding theory is denoted $\mathcal{E}\ell\ell_{E,\mathcal{T}}^*$. Even though we call this an 'equivariant' cohomology theory, it is a just a cohomology theory in the usual sense: the spaces on which this cohomology theory is defined are not equipped with any kind of action.

We will not enter into the details of the finiteness assumptions that $\mathcal{T}$ should satisfy. All that we'll need is the assumption that the Drinfel'd center of $\mathcal{T}$ is a modular tensor category. The $\mathcal{T}$-equivariant elliptic cohomology $\mathcal{E}\ell\ell_{E,\mathcal{T}}$ is a module over the non-equivariant elliptic cohomology $\mathcal{E}\ell\ell_E^*$, and satisfies

$$\mathcal{E}\ell\ell_{E,\mathcal{T}}^*(X) = \mathcal{E}\ell\ell_{E,\mathcal{T}}^*(pt) \otimes_{\mathbb{Q}} H^*(X,\mathbb{Q}) \qquad \text{and}$$
$$\mathcal{E}\ell\ell_{E,\mathcal{T}}^*(pt) = \mathcal{E}\ell\ell_{E,\mathcal{T}}^0(pt) \otimes_{\mathcal{E}\ell\ell_E^0(pt)} \mathcal{E}\ell\ell_E^*(pt) = \mathcal{E}\ell\ell_{E,\mathcal{T}}^0(pt)[\omega^{\pm 1}]$$

By the above formulae, in order to define the cohomology theory $\mathcal{E}\ell\ell_{E,\mathcal{T}}^*$, it is enough to describe the k-module $\mathcal{E}\ell\ell_{E,\mathcal{T}}^0(pt)$. We will define the latter so as to only depends on the elliptic curve E and the modular tensor category $Z(\mathcal{T})$.

2 Reshetikhin-Turaev state spaces for elliptic curves over rings

Another way of describing the goal of this paper is that it produces a generalization of Reshetikhin-Turaev state spaces.

Given a modular tensor category $\mathcal{C}$ over $\mathbb{C}$ and a compact oriented surface Σ, the Reshetikhin-Turaev construction associates to the above data a complex vector space $RT_{\mathcal{C}}(\Sigma)$. We generalize this to a setup where the surface Σ is replaced by an elliptic curve E over some ring k of characteristic zero. When $k = \mathbb{C}$, our construction recovers the usual Reshetikhin-Turaev state space. This new Reshetikhin-Turaev state space $RT_{\mathcal{C}}(E)$ is a k-module, it is one and the same thing as the generalized equivariant elliptic cohomology discussed in the previous section:

$$RT_{Z(\mathcal{T})}(E) = \mathcal{E}\ell\ell^0_{E,\mathcal{T}}(pt).$$

Overall goal: Given a modular tensor category $\mathcal{C}$ over the complex numbers, and an elliptic curve E over a ring k of characteristic zero, we want to construct a k-module

$$RT_{\mathcal{C}}(E).$$

Construction: Let r be the rank of $\mathcal{C}$ (the number of simple objects). Let n be an integer such that the representation $SL(2,\mathbb{Z}) \to GL(r, \mathbb{Q}_{ab})$ given by the (normalised) modular S and T matrices factors as

$$\begin{array}{ccc} SL(2,\mathbb{Z}) & \longrightarrow & GL(r,\mathbb{Q}_{ab}) \\ \downarrow & & \uparrow \\ SL(2,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & GL(r,\mathbb{Q}[\zeta_n]). \\ \in & & \in \\ A & \longmapsto & M(A) \end{array}$$

Here, ζ_n is a primitive nth root of unity, and $\mathbb{Q}_{ab} = \bigcup_n \mathbb{Q}[\zeta_n]$.

Let $E[n]$ be the group scheme of n-torsion points of E. By étale descent, it is enough to define $RT_{\mathcal{C}}(E)$ when the étale map $E[n] \to \mathrm{Spec}(k)$ is trivial (isomorphic to $\mathrm{Spec}(k) \times \mathbb{Z}/n\mathbb{Z}$). If $E[n]$ is not trivial, we let $\mathrm{Spec}(k)^{[n]}$ be the scheme whose S-points consist of a point $x : S \to \mathrm{Spec}(k)$ and an isomorphism $(\mathbb{Z}/n\mathbb{Z})^2 \cong E[n]_x$, where $E[n]_x$ is the set of lifts $S \to E[n]$ of x. We then define

$$RT_{\mathcal{C}}(E) := RT_{\mathcal{C}}\Big(E \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k)^{[n]}\Big)^{GL(2,\mathbb{Z}/n\mathbb{Z})}.$$

We have therefore reduced the problem of defining $RT_{\mathcal{C}}(E)$ to the special case when $E[n] \to \mathrm{Spec}(k)$ is a trivial bundle (this implies in particular that k has enough nth roots of unity).

This is where the construction really starts:

Pick a trivialisation $\varphi : (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{\cong} E[n]$, and define

$$RT_{\mathcal{C}}(E) \;:=\; k^r.$$

Given two trivialisations $\varphi_1, \varphi_2 : (\mathbb{Z}/n\mathbb{Z})^2 \to E[n]$, we need to provide an isomorphism

$$f_{12} : k^r \to k^r.$$

Let

$$A_{12} := \varphi_2^{-1}\varphi_1 \qquad \text{and} \qquad \overline{A}_{12} := \left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} A_{12} \in SL(2, \mathbb{Z}/n\mathbb{Z}).$$

The map

$$\det(\varphi_1) \;:\; \mathbb{Z}/n\mathbb{Z} \to \det E[n] \cong \mu_n$$

(where the isomorphism $\det E[n] \cong \mu_n$ comes from the Weil pairing) induces a field homomorphism

$$\alpha_1 : \mathbb{Q}[\zeta_n] \to k.$$

We define

$$f_{12} := \alpha_1(M(\overline{A}_{12})) \in GL(r, k).$$

For the above definition to be consistent, we need to verify that given three isomorphisms $\varphi_1, \varphi_2, \varphi_3 : (\mathbb{Z}/n\mathbb{Z})^2 \to E[n]$ the cocycle condition $f_{23} \circ f_{12} = f_{13}$ holds.

We have

$$\begin{aligned} f_{23} \circ f_{12} &= \alpha_2(M(\overline{A}_{23}))\alpha_1(M(\overline{A}_{12})) \\ &= \alpha_2\Big(M\Big(\left(\begin{smallmatrix} \det(A_{23}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} A_{23}\Big)\Big)\alpha_1\Big(M\Big(\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} A_{12}\Big)\Big) \end{aligned}$$

and

$$f_{13} = \alpha_1\Big(M\Big(\left(\begin{smallmatrix} \det(A_{13}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\Big)A_{13}\Big) \in GL_2(r, k).$$

Apply α_1^{-1} to both expressions:

$$\begin{aligned} \alpha_1^{-1}(f_{23} \circ f_{12}) &= \alpha_1^{-1}\alpha_2\Big(M\Big(\left(\begin{smallmatrix} \det(A_{23}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} A_{23}\Big)\Big)M\Big(\overline{A}_{12}\Big) \\ \alpha_1^{-1}(f_{13}) &= M\Big(\left(\begin{smallmatrix} \det(A_{13}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} A_{13}\Big) \\ &= M\Big(\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} \det(A_{23}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} A_{23}\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)\overline{A}_{12}\Big) \\ &= M\Big(\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\overline{A}_{23}\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big)M\Big(\overline{A}_{12}\Big). \end{aligned}$$

So we are reduced to checking the following equation:

$$\alpha_1^{-1}\alpha_2\Big(M\Big(\overline{A}_{23}\Big)\Big) \stackrel{?}{=} M\Big(\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\overline{A}_{23}\left(\begin{smallmatrix} \det(A_{12}) & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big).$$

Recall that $\alpha_1^{-1}\alpha_2 = \det(\varphi_1^{-1}\varphi_2) : \mathbb{Z}/n\mathbb{Z} \to Z/n\mathbb{Z}$ is multiplication by $\det(A_{12})^{-1}$. The above equation follows from a general fact about modular S and T matrices:

Lemma 1. *If $u \in (\mathbb{Z}/n\mathbb{Z})^\times$ (i.e. $u \in \mathbb{Z}/n\mathbb{Z}$ is coprime to n) and M is the representation of $SL(2, \mathbb{Z}/n\mathbb{Z})$ coming from a modular category, then we have*

$$\sigma_u\big(M(A)\big) = M\left(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) A \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\right).$$

Here, $\sigma_u \in \mathrm{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ denotes the Galois automorphism associated to $u \in (\mathbb{Z}/n\mathbb{Z})^\times$.

Proof. Both sides are multiplicative in A, since $\sigma_u(M(AB)) = \sigma_u(M(A))\sigma_u(M(B))$ and

$$\begin{aligned} M\left(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) AB \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\right) &= M\left(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) A \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) B \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\right) \\ &= M\left(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) A \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\right) M\left(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) B \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\right) \end{aligned}$$

(even though $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \notin SL(2, \mathbb{Z}/n\mathbb{Z})$).

So we only need to verify the identity on the generators $s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $SL(2, \mathbb{Z}/n\mathbb{Z})$. We write $S = M(s)$ and $T = M(t)$. These then reduce to a couple of known facts about modular data.

For $A = s$, we note that

$$\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & -u \\ u^{-1} & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),$$

and that both of these matrices are in $SL(2, \mathbb{Z}/n\mathbb{Z})$. Let $G_u := M\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ (the matrices G_ℓ for $\ell \in (\mathbb{Z}/n\mathbb{Z})^\times$ are the signed permutation matrices realizing the action of the Galois group on the simple objects of the modular category). We have

$$M\big(\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\big) = G_u S,$$

and as the Galois group actions on the simple objects and on the field are related by the formula $G_\ell S = \sigma_\ell(S)$ [1, Prop 2.2], we have the desired result.

For $A = t$, the computation is simpler:

$$\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) = t^u,$$

so $M(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) t \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}) = T^u$. Since T is diagonal with diagonal entries which are n-th roots of unity, we also have $\sigma_u(T) = T^u$. □

Questions:

• If one takes $\mathcal{T} = Vec[G]$ for the finite group $G = \mathbb{Z}/m\mathbb{Z}$, in other words, if one takes $\mathbb{C} = Vec_G[G]$, then the resulting RT k-module should be $\mathcal{O}_{E[m]}$.

• If one takes $S = (1)$ and $T = (\zeta_3)$, then the resulting RT k-module should be $\omega^{\otimes 4}$ (recall that $\omega^{\otimes 12}$ is canonically trivial over $\mathcal{M}_{1,1}^{\mathbb{Q}}$).

• Fix the elliptic curve E. Show that the construction $\mathcal{T} \mapsto RT_{Z(\mathcal{T})}(E)$ extends to a functor. Given a bimodule category ${}_{\mathcal{T}_1}\mathcal{M}_{\mathcal{T}_2}$, there should be an associated linear map $RT_{Z(\mathcal{T}_1)}(E) \to RT_{Z(\mathcal{T}_2)}(E)$.

• Fix $\mathcal{C}$. And look at $E \mapsto RT_{\mathcal{C}}(E)$: this is a rank r vector bundle over the moduli stack of elliptic curves $\mathcal{M}_{1,1}^{\mathbb{Q}}$. One can try to compute its global sections. That's a module over the ring $\mathbb{Q}[g_2, g_3]$ of modular forms. What is that module?

References.

[1] Terry Gannon and Scott Morrison, *Modular data for the extended Haagerup subfactor*, ArXiv: 1606.07165 (2016).