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Subfactors arising from positive energy representations of some infinite–dimensional groups

Preliminary notes by Antony WASSERMANN

Abstract — To each irreducible positive energy representation of the loop group of a simple compact Lie group we associate canonically an irreducible subfactor of the hyperfinite factor R_1 of Type III_1. These are parametrised by the same data as the Jones–Wenzl subfactors of the hyperfinite II_1 factor R and we conjecture that the loop group subfactors are tensor products of these subfactors with R_1. We also define subfactors of R_1 for each discrete series representation of Diff S^1 and conjecture that these have finite index. We give preliminary evidence supporting these conjectures. In particular the subfactors for LSU(2) at level 2 or for Diff S^1 with central charge 1/2 have index either 1 or 2, and those for LSU(2) at level 4 or LSU(4) at level 2 have finite index.

1. INTRODUCTION. — Quantum mechanics and group representation theory were two of the main reasons for von Neumann to invent rings of operators in 1934. Quantum field theory was developed over the next fifty years in a number of different ways — constructive, axiomatic, algebraic and through functional integrals. In the last ten years two–dimensional conformally invariant theories have become objects of interest in both mathematics and physics. They are being studied from many different points of view. Our starting point will be their connection with positive energy representations of certain infinite–dimensional groups. We hope to use von Neumann algebras and the theory of subfactors to understand some aspects of conformal field theory.

In this note we shall be concerned with the connections between three different areas in mathematics and physics: exactly solvable lattice models in statistical mechanics, conformal field theory in two space–time dimensions and the theory of subfactors. For some time it has been known that the 'Yang–Baxter' braiding in critical lattice models can be used to define subfactors. On the other hand the continuum limit of a critical statistical mechanical system should provide a conformal field theory (CFT) with the monodromy of a chiral part of the CFT reproducing the braiding associated to the original system. This link has not been established so far, except in a few special cases. However Tsuchiya and Kanie [TK] discovered by direct computation that the monodromy for the Wess–Zumino–Witten models for $SU(2)$ (and more generally $SU(N)$) agreed with the braiding of restricted solid–on–solid models, a class of critical models studied by Jimbo, Miwa, et al. (For $SU(2)$ these are the Andrews–Baxter–Forrester models and the subfactors are those originally found in [Jones]). So it was natural to ask whether there was a more fundamental way of producing subfactors from a conformal field theory that did not directly invoke the braiding.

Such a construction of subfactors is obtained by considering localised fields, in our case the von Neumann algebras generated by local subgroups of loop groups or diffeomorphism groups. Algebraic quantum field theory ([DHR], [FRS]) provide a framework for studying abstract local algebras with 'Haag duality' playing a pivotal rôle. We have had to adapt this theory to our concrete models. Most of the basic postulates must be proved as theorems. An important aspect of our approach, which marks a departure from algebraic QFT, is that we may crucial use of interrelationships between different theories. This stems from the natural hierarchy of CFT. Using a key result of Takesaki [T] relating modular groups of von Neumann algebras and their subalgebras, we may descend from relatively simple theories, such as free field theories, to more complicated subtheories.

2. JONES–WENZL SUBFACTORS AND SOLVABLE LATTICE MODELS. — It has been known for some time that subfactors could very naturally be associated with certain statistical mechanical models. More accurately, the reverse happened: in [Jones] it was shown that all subfactors of index less than four canonically contain an algebra generated by projections e_i satisfying $e_ie_{i\pm1}e_i\tau e_i$ and $e_ie_j = e_je_i$ ($|i-j| > 1$), where τ^{-1} equals the

index. This index must have the form $4\cos^2(\pi/\kappa)$ for $\kappa = 3, 4, 5, \ldots$ It was only later realised these e_i's were the transfer matrices for the Andrews–Baxter–Forrester model. More significantly for us, the e_i's themselves give rise to a subfactor with the same index simply by taking the subalgebra obtained by omitting the first generator e_1.

In general exactly solvable lattice models come equipped with R–matrices $R \in GL(V \otimes V)$ and density matrices $a \in End(V)$. The R–matrix satisfies the braid relation $R_1R_2R_1 = R_2R_1R_2$ in $GL(V \otimes V \otimes V)$, a version of the quantum Yang–Baxter equation. (Here and below R_i lives in the i and $i+1$ copies of V.) Let $End(V)^{\otimes\infty}$ be the algebra generated by tensors $x_1 \otimes x_2 \otimes \cdots$ with $x_i = 1$ eventually. The map $\pi(\sigma_i) = R_i$ defines a representation of the infinite braid group B_∞ in $End(V)^{\otimes\infty}$ and the tensor powers of a define a form on $End(V)^{\otimes\infty}$ which restricts to a trace tr on $\pi(\mathbb{C}B_\infty)$. These have the properties: (a) $\dim \pi(\mathbb{C}B_n) < \infty$ for all n; (b) $\mathrm{tr}(w\sigma_n^{\pm1}) = \mathrm{tr}(w)\mathrm{tr}(\sigma_n^{\pm1})$ for $w \in B_n$. Typically the models and their matrices R and a depend on one or more parameters. For special choices of parameter, usually roots of unity, the algebras $\pi(\mathbb{C}B_n)$ fail to be semisimple. Nevertheless for these exceptional values tr defines a positive trace on $\mathbb{C}B_\infty$. So we may construct $L^2(B_\infty, \mathrm{tr})$, the Hilbert space completion of $\mathbb{C}B_\infty$ with inner product $tr(ab^{-1})$. Let π be the representation of B_∞ by left translation. As Wenzl has shown, this has a very strong finiteness property: (c) $\dim Z(\pi(\mathbb{C}B_n))$ is uniformly bounded. The shift ρ given by $\rho(e_i) = e_{i+1}$ induces an endomorphism of the hyperfinite type II_1 factor $M = \pi(\mathbb{C}B_\infty)'$ into itself. Starting with the inclusion $N = \rho^n(M) \subset M$, Wenzl obtains his subfactors by taking the reduction $Np \subset pMp$ by minimal projections $p \in N' \cap M$.

The specific R–matrices used are those coming from the identity representations of quantum groups at roots of unity; they probably arise from the restricted solid–on–solid models of Jimbo et al. Data about these subfactors is encoded in $K_0(\pi(\mathbb{C}B_\infty))$ which has a natural multiplication making it a 'fusion algebra' [GW]. These subfactors can be regarded as quantum versions of the subfactors obtained from classical invariant theory. Taking the product type action of a compact group $G \subset GL(V)$ on $\mathrm{End}V^{\otimes\infty}$, one obtains a factor from the fixed point algebra and the subfactor from the shift and the fusion algebra is roughly $R(G)$. The Jones–Wenzl subfactors correspond to the fixed point algebra of a quantum deformation of G with the shift. They are usually described by tangle algebras and their study is part of quantum invariant theory.

3. POSITIVE ENERGY REPRESENTATIONS — Let G be a connected simple compact Lie group. The loop group LG consists of all smooth maps from the circle G and the group of orientation preserving diffeomorphisms $\mathrm{Diff}(S^1)$ acts by automorphisms on LG. A positive representation is a homomorphism $LG \to PU(H)$ which extends to $LG \rtimes \mathrm{Rot}(S^1)$ in such a way that the energy operator, the infinitesimal generator L_0 the rotation group $\mathrm{Rot}(S^1)$, is positive. The central extension $U(H)$ of $PU(H)$ by $\mathbb{T}$ induces a central extension $\tilde{L}G$ of LG which is characterised by a positive integer ℓ called the *level*. If H is irreducible, the eigenspaces for $\mathrm{Rot}(S^1)$ are finite–dimensional and invariant under the subgroup G of constant loops. In particular the lowest energy subspace $H^0 = V \subset H$ is an irreducible G–module. The pair (ℓ, V) uniquely determines the representation and for a given level only finitely many V in $\hat{G}$ can occur. Any positive energy representation extends uniquely to $LG \rtimes \mathrm{Diff}(S^1)$, so that it is invariant under reparametrisation. The resulting projective representation of $\mathrm{Diff}(S^1)$ determines a central extension of $\mathrm{Diff}(S^1)$. This is characterised on the infinitesimal level by a number $c > 0$, the *central charge*, and globally by some finite covering of the circle (a rational number $h \in [0,1)$). Positive energy representations of Diff S^1 with $c \in (0,1)$ are also uniquely specified by the lowest eigenvalue $h/ge0$ of L_0 (which is simple). The possible values for c in this range are $c = 1 - 6/(m+2)(m+3)$ ($m \geq 1$) and the corresponding values of h are given by $h = [((m+3)p - (m+2)q)^2 - 1]/4(m+2)(m+3)$ where $1 \leq p \leq qlem+1$. If π is

is G represented in U(H) or PU(H)?

$h \geq 0$

$q \leq m+1$

q^{-c}

$q \uparrow 1 \Leftrightarrow t \downarrow 0$, $\frac{1}{t} \uparrow \infty$

$\mathrm{Tr}_\pi\, q^{L_0} = q^{c/24}\, \mathrm{Tr}_\pi\, q^{L_0 - c/24} = q^{c/24}\chi_\pi(\tau)$
$= q^{c/24} S_{\pi\pi'}\chi_{\pi'}(-1/\tau)$
$\approx q^{c/24} S_{\pi 0}(e^{-2\pi i/\tau})^{0-c/24}$
$\approx a(\pi)\, e^{\frac{2\pi c}{12 t}}$
where $-2\pi t = \log q$
$= a(\pi) \exp\left(-\frac{(2\pi)^2 c}{12 \log q}\right)$

a positive energy representation and $z \to z^{L_0}$ denotes its restriction to (the finite cover of) the circle group, then z^{L_0} can be analytically continued to the complex plane. When $|z| < 1$, the operators z^{L_0} is a trace class contraction and for $0 < q < 1$, we have $\mathrm{Tr}(q^{L_0}) \sim a(\pi)\exp(-1/12\log q)$ as $q \uparrow 1$ where $a(\pi) > 0$ is the *asymptotic dimension* of π [KW]. $(2\pi)^2 c$

If H is a subgroup of G, not necessarily semisimple, and π is an irreducible positive energy representation of LG on $\mathcal{H}$, then the restriction of π to LH is also of positive energy so breaks up as direct sum of irreducibles for a given cocycle of LH. If there are only finitely many summands, we have a conformal inclusion. This can only occur when the central charges c_G and c_H for G and H are equal, and then the level of π must be 1. The complete list of all conformal inclusions is known ([Bais-Bouwknegt], [Schellekens-Warner]) together with almost all the corresponding branching rules. In particular when G is simply laced with maximal torus T, the inclusion $LT \subset LG$ at level one is conformal and the restrictions to LT remain irreducible and inequivalent. In the next section we use Fermi-Dirac fields to understand these representations. We shall discuss other examples when we need them.

The finite reducibility theorem does not apply to the identity component of the torus subgroup, only to the full torus subgp.

When the inclusion is not conformal, then we can obtain new representations of $\mathrm{Diff}(S^1)$ on the multiplicity spaces of LH following [GKO] (see also Lemma 13.9 in [Segal]). This *coset construction* yields representations with central charge $c = c_G - c_H$. Indeed we may write $H = \oplus H_i \otimes K_i$ where the H_i's are distinct irreducible representations of LH and the K_i's are multiplicity spaces. So $\pi(LH)' = \oplus \mathbb{C} \otimes B(K_i)$. Now fixing a diffeomorphism f, we suppose it acts as $\mathrm{Ad}(U_f)$ on H and $\mathrm{Ad}(V_i)$ on each H_i. Let $V_f = \oplus V_i \otimes I$ on H. Then since $LH \subset LG$, $\mathrm{Ad}(U_f)$ and $\mathrm{Ad}(V_f)$ satisfy $\mathrm{Ad}(U_f\pi(h)U_f^*) = \mathrm{Ad}(V_f\pi(h)V_f^*)$ for all $h \in LH$. Thus $W_f = V_f^* U_f$ lies in $\pi(LH)'$. The restrictions of $\mathrm{Ad}(W_f)$ to each K_i are projective representations of $\mathrm{Diff}(S^1)$. We shall only use the case $SU(2)_{n-1} \times SU(2)_1 \supset SU(2)_n \times \mathrm{Diff}(S^1)_n$ in this paper.

Given all the level one representations, all the level ℓ representations of LG can be obtained by decomposing their ℓ-fold tensor products. The level one representations can be obtained by two explicit 'free field' models when G is simply laced. On the one hand the Clifford algebra and free fermions give rise to the level one representations of the unitary and orthogonal groups. On the other hand when G is simply laced and $T \subset G$ is a maximal torus, $LT \subset LG$ is a conformal inclusion and the level one representations of LG restrict to irreducible representations of LT. But these are essentially described by free bosons and the extension to LG is obtained at the Lie algebra level by means of vertex operators (Frenkel-Kac-Segal [GO]). $\psi M \psi$ $e^{i\alpha\theta}$

4. Local subgroups and their subfactors. — Let $\pi : LG \to PU(\mathcal{H})$ be a positive energy irreducible representation of the loop group LG on the Hilbert space $\mathcal{H}$. If I is an open interval on the circle, we let the L_IG be the (normal) subgroup of LG consisting of loops equal to the identity outside I. Let I^c denote the open interval complementary to I. Then if G is simply connected, $\pi(\gamma_1)\pi(\gamma_2) = \pi(\gamma_2)\pi(\gamma_1)$ (this doesn't really make sense) (Pass to $\mathcal{U}_{univ}$) for $\gamma_1 \in L_IG$ and $\gamma_2 \in L_{I^c}G$. Thus the von Neumann algebra $N = \pi(L_IG)''$ generated on $\mathcal{H}$ by L_IG is contained in $M = \pi(L_{I^c}G)'$, the commutant of $L_{I^c}G$ on $\mathcal{H}$. We ~~have~~ show below that N and M are hyperfinite type III_1 factors, so that the inclusion $N \subseteq M$ gives a subfactor. Haag duality is said to hold when $N = M$. *Thus the subfactor measures the failure of Haag duality in non-vacuum sectors.* A similar construction applies to discrete series representations of $\mathrm{Diff}\, S^1$. The rôle of L_IG is played by $\mathrm{Diff}_I(S^1)$, the subgroup of diffeomorphisms fixing points outside I. In general we conjecture that a positive energy representation π should lead to a subfactor of R_1 ofindex $[a(\pi)/a(\pi_0)]^2$ where π_0 is the vacuum representation. More specifically we conjecture that the loop group subfactors, which are labelled by the same data as Jones-Wenzl subfactors, are in fact obtained from $a(\pi) = S_{\pi 0}$

them by tensoring with the hyperfinite III_1 factor.

To study these subfactors it has been useful to introduce another class of subfactors associated with conformal or GKO inclusions. For example if π is a level one positive energy representation of L_G and $LH \subset LG$ is a conformal inclusion, then $\pi(L_IH)'' \subset \pi(L_IG)''$ gives a subfactor. These are in some sense 'square roots' of the class above: applying the basic construction of [Jones] to them we recapture the first type of subfactor. Note that for a product $G = G_1 \times G_2$, we have $\pi_1 \otimes pi_2(j(L_IG))'' = \pi_1(L_IG_1)'' \otimes \pi_2(L_IG_2)''$ where $j : LG \to LG_1 \times LG_2$ is the map $j(f)(z) = (p_1f(z), p_2f(z))$ determined by the projections $p_i : G \to G_i$. see §6 $\pi(L_IG)'' \subset \pi(L_{I^c}G)'$

"Takesaki devissage"

5. Tomita-Takesaki theory. — Suppose that the vector ξ in H is cyclic for both the von Neumann algebra M and its commutant M'. The fundamental operators S and F of Tomita are defined as the closures of the conjugate linear operators $S(x\xi) = x^*\xi$ on $M\xi$ and $F(y\xi) = y^*\xi$ on $M'\xi$. These operators are each other's adjoints and S has polar decomposition $S = J\Delta^{1/2}$ where $\Delta = S^*S$ and J is a conjugate linear isometry satisfying $J^2 = I$. Moreover conjugation by Δ^{it} defines one-parameter automorphism groups σ_t of both M and M' and we have $M' = JMJ$. These results apply also when $\mathcal{H}$ and M are $\mathbb{Z}_2$-graded and ξ is even. The *Klein transformation* K on $\mathcal{H}$ is defined to be 1 and i on the even and odd parts of $\mathcal{H}$ respectively: the graded commutant of M is just $KM'K^{-1}$.

The Kubo-Martin-Schwinger condition is useful for verifying that a given σ_t is the modular automorphism group of a state φ: if b is a vector in M for which $t \to \sigma_t(b)$ has an extension to an entire function then one just has to check that $\varphi(ab) = \varphi(\sigma_i(b)a)$ for a dense set of a in M.

For descending to subtheories, we shall need an additional result of Takesaki [T]. Let N be a von Neumann subalgebra of M invariant under σ_t. Then σ_t and J restrict to the modular automorphism group and conjugation operator of N for ξ on the closure H_1 of $N\xi$. Moreover $H_1 = H$ if and only if $M = N$. Finally there is a unique normal conditional expectation of M onto N consistent with σ_t and preserving the vector state.

In practice the verification that a vector is cyclic can either be seen directly or through a simple version of the Reeh-Schlieder theorem. Let U_z be a unitary representation of the circle group on H of positive energy and let M be a von Neumann algebra acting on H such that the operators $U_zXU_z^*$ ($X \in M$, $z \in S^1$) act irreducibly on H. If ξ is any eigenvector of U_z and N is a von Neumann algebra containing $U_zMU_z^*$ for z sufficiently close to 1, then ξ is cyclic for N.

Finally we recall that if the modular group for ξ is ergodic on M, then M must be a factor of type III_1 [Connes]; this will be the case whenever ξ is the only vector in $\ker\Delta$.

6. Duality for CAR algebras. — If H is complex Hilbert space, $\mathrm{Cliff}(H)$ is the $\mathbb{Z}_2$-graded C^* algebra generated by operators $a(f)$ satisfying the canonical anticommutation relations (CAR) $[a(f), a(g)]_+ = 0$ and $[a(f), a(g)^*]_+ = (f, g)I$. ($a$ is thus a complex fermi field.) If $0 \le A \le I$ is an operator on H, we get a quasifree factor state φ_A on $CAR(H)$ through the formula: $\varphi_A(a(g_n)^* \cdots a(g_1)^* a(f_1) \cdots a(f_m)) = \delta_{mn}\det((Af_i, g_j))$. ϕ_A is pure iff A is a projection. Two states ϕ_A and ϕ_B belong to the same representation if and only if $A - B$ is Hilbert-Schmidt.

The unitary group $U(H)$ acts functorially by automorphisms on $\mathrm{Cliff}_D(H)$. If P has infinite rank and corank (i.e. is in general position), the restricted unitary group

$$U_{\mathrm{res}}(H) = \{U \in U(H) : [P, U]\,\text{Hilbert-Schmidt}\}$$

is the subgroup of $U(H)$ taking ϕ_P onto equivalent states. If $\mathcal{H}_P$ is the $\mathbb{Z}_2$-graded Hilbert space obtained by the GNS construction from ϕ_P, then there is a unique projective representation of $\pi : U_{\mathrm{res}}(H) \to PU(H_P)$ compatible with the action of $CAR(H)$.

The Hilbert space $\mathcal{H}_P$ is realised through the 'Dirac sea' construction of fermionic Fock space. We take $\mathcal{H}_P = \Lambda(PH \hat{\otimes} (P^{\perp} H)^*)$. The action of $a(f)$ is the sum of exterior multiplication by Pf on the first component (creation) and contraction by $P^{\perp} f$ on the second component (annihilation). The vacuum vector Ω gives the state φ_P.

The map $U_{\mathrm{res}}(H) \to PU(H_P)$ is continuous for the topology on $U_{\mathrm{res}}(H)$ given by the strong operator topology together with the metric topology $d(U, V) = \|[P, U - V]\|_2$. Let α be the conjugate-linear *-automorphism of $\mathrm{Cliff}(H)$ defined by $\alpha(a(f)) = a(f)^*$. Note that if $U \in U(H)$ is such that $UPU^* = P^{\perp}$ (modulo Hilbert–Schmidt operators), then the conjugate-linear *-automorphism $\alpha \cdot \alpha_U$ extends by continuity to $B(\mathcal{H}_P)$. It is therefore implemented by an essentially unique antiunitary on $\mathcal{H}_P$.

We shall need the following version of Haag–Araki duality, implicit in the literature. Let Q be a projection in H such that P and Q are in general position, i.e. neither Q nor $Q^{\perp}$ meet P or $P^{\perp}$. Then the graded commutant B of $A = CAR(QH)''$ in H_P is $CAR(Q^{\perp} H)''$. The modular automorphism group corresponding to ξ_P is implemented by $\mathrm{Cliff}(H^{it})$ where $\Delta = QPQ/QP^{\perp}Q + Q^{\perp}P^{\perp}Q^{\perp}/Q^{\perp}PQ^{\perp}$. Both A and B are isomorphic to R_1 as $U_t = \mathrm{Cliff}(\Delta^{it})$ is ergodic on A and B.

Firstly we show that the vacuum vector Ω is cyclic for $\mathrm{Cliff}(QH)$, i.e. the closure $\mathcal{H}_0$ of $\pi_P(\mathrm{Cliff}(QH))\Omega$ is $\mathcal{H}_P$. Assume by induction that forms of degree N or less lie in $\mathcal{H}_0$, and take ω an N-form and $f \in QH$. Then $a(f)\omega \equiv Pf \wedge \omega$ modulo Λ^N and $a(f)^*\omega = (P^{\perp} f)^* \wedge \omega$ modulo Λ^N, and therefore $Pf \wedge \omega, (P^{\perp} f) \wedge \omega \in H_0$. We get any $N+1$-form using the density of PQH and $P^{\perp}QH$ in PH and $P^{\perp}H$. Applying this result to $Q^{\perp}$, we see that Ω is also separating.

We nest check that $\sigma_t = \mathrm{AD}\, U_t$ satisfies the KMS condition for the vacuum state restricted to $\mathrm{Cliff}(QH)$. Note that φ_P restricts to φ_A where $A = QPQ$. If $f \in \mathcal{H}$ is an entire vector for U_t, then $\sigma_i(a(f)) = a(Bf)$ where $B = A/I - A$. Let M_1 be the weakly dense *-subalgebra generated by the products $x = a(g_n)^* \cdots a(g_1)^* a(f_1) \cdots a(f_m)$ with the f_i's and g_j's are entire. Applying the anticommutation relations, we obtain $\varphi_A(xa(f)) = -\varphi_A(a(f)x) + \varphi_A(xa(A^{-1}f))$. Replacing f by $(I - A^{-1})f$ (f is entire), we obtain $\varphi_A(xa(f)) = \varphi_A(a(Bf)x) = \varphi_A(\sigma_i(a(f))x)$ where $B = A/I - A$. But the $a(f)$'s generate M_1, so the modular condition follows.

The proof may be completed in two ways. For the first we note that $\pi_P(\mathrm{Cliff}(Q^{\perp}H))^g$ has Ω as a cyclic separating vector and σ_t as modular group. Since it contains $\pi_P(\mathrm{Cliff}(Q^{\perp}H))''$, which is invariant under σ_t and cyclic, we obtain equality by Takesaki's theorem. For the second more direct proof, we use the unbounded operator $T : Pf \mapsto iP^{\perp}f, P^{\perp}g \mapsto iPg$ for $f \in QH$, $q \in Q^{\perp}H$. Then T has polar decomposition $U\Delta$ and, if $f \in QH$ is entire, we have $Ja(f)\Omega = S\Delta^{-1}a(f)\Omega = ia(Uf)^*\Omega$. Since Uf lies in $Q^{\perp}H$, $a(Uf)^*$ lies in the graded commutant. Since the $a(f)$'s generate $\mathrm{Cliff}(QH)$, we obtain $J = K \cdot \mathrm{Cliff}U$ where K is the Klein transformation and the result follows.

We explain T more carefully (we shall need it below). The Hilbert space $PH \oplus P^{\perp}H$ can be identified with H. The subspace QH then defines the graph of a closed densely defined operator from PQH to $P^{\perp}QH$, taking Pf to $P^{\perp}f$ for $f \in QH$. Using this observation we define a closed densely defined operator $T(P, Q)$ on H by sending Pf to $iP^{\perp}f$ for $f \in QH$ and sending $P^{\perp}f$ to iPf for $f \in Q^{\perp}H$. This operator is self-adjoint and its graph in $H \oplus H$ has the following projection in $M_2(B(H))$:

$$\begin{pmatrix} PQP + P^{\perp}Q^{\perp}P^{\perp} & i(PQ - QP) \\ i(QP - PQ) & P^{\perp}QP^{\perp} + PQ^{\perp}P \end{pmatrix}$$

From this expression, we see immediately that $T(P, Q) = -T(Q, P) = -T(P^{\perp}, Q)$. Note that $TP = P^{\perp}T$ and $TQ = Q^{\perp}T$. Let $T = U \cdot \Delta^{1/2}$ be the polar decomposition of T. Then $UPU^* = P^{\perp}$, $UP^{\perp}U^* = P$, etc. Moreover $\Delta = T^*T = QPQ/QP^{\perp}Q \oplus Q^{\perp}P^{\perp}Q^{\perp}/Q^{\perp}PQ^{\perp}$.

These results extend easily to real fermions (see [A]). The self dual CAR algebra $\mathrm{Cliff}_{\Gamma}(H)$ is defined for a complex Hilbert space H with a conjugate linear involution Γ. It is generated by operators $a(f)$ subject to $a(\Gamma f) = a(f)^*$ and $[a(f), a(g)^*]_+ = (f, g)I$. If P and Q are projections in general position such that $\Gamma P \Gamma = P^{\perp}$ and $\Gamma Q = Q\Gamma$, then we define $\mathcal{H}_P = \Lambda(PH)$ with $a(f)$ acting by the sum of exterior multiplication by Pf and contraction by $P(\Gamma f)^*$. The von Neumann algebras generated by $\mathrm{Cliff}_M(QH)$ and $\mathrm{Cliff}_M(Q^{\perp}H)$ are each others quasicommutants and the modular groups are implemented by the same operators.

~~7. THE GEOMETRIC MODULAR GROUP FOR FERMIONS ON THE CIRCLE.~~ — Let $H = L^2(S^1) \otimes V$ where V is a finite-dimensional inner product space. Let P be the orthogonal projection onto $H^2(S^1) \otimes V$, the Hardy space of boundary values of holomorphic (vector-valued) functions on the disc and let Q be the orthogonal projection onto $L^2(I) \otimes V$ where I is the upper half of the circle. P and Q are in general position, since no non-zero holomorphic or antiholomorphic function can have boundary values vanishing in an interval. We compute the modular group in this case in two ways, using 2×2 matrices and by analytic continuation. Both calculations are at the prequantised level using the formula for Δ (cf [BW], [BS-M] for quantised proofs). The modular group corresponds to the Möbius transformation group on the circle that fixes ± 1 and modular conjugation to the orientation reversing reflection $z \mapsto z^{-1}$.

(1) The unitary equivalence between $L^2(\mathbf{T})$ and $L^2(\mathbf{R})$ induced by the Cayley transform $z \mapsto i(z+1)/(z-1)$ carries the Möbius flow on the circle fixing ± 1 onto the scaling action of $\mathbf{R}_+^*$ on $\mathbf{R}$ fixing 0 (and ∞). It also carries Q onto the characteristic function of the positive half-line and P onto projection onto the Hardy space of boundary values of holomorphic functions on the upper half plane. If $V : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ denotes the Fourier transform, then $P = VQV^*$.

Let $\Delta = QPQ/QP^{\perp}Q$: we have to compute Δ^{it}. For $f \in L^2(\mathbf{R})$ define $f_{\pm} \in L^2(\mathbf{R})$ by $f_{\pm}(t) = e^{t/2} f(\pm e^t)$ and set $W(f) = (\hat{f}_+, \hat{f}_-)$. Thus W is an unitary between $L^2(\mathbf{R})$ and $L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$. For any operator T on $L^2(\mathbf{R})$ let $T_s = WTW^*$ be its image under the transform W. Clearly $Q_s = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The scaling action of $\mathbf{R}_+^*$ on $L^2(\mathbf{R})$ is given by $(\pi(s)f)(x) = \sqrt{s} \cdot f(sx)$ and becomes the (diagonal) multiplication operator for s^{ix} under W. Because this representation has muliplicity two, the von Neumann algebra of multiplication operators $\mathcal{A}$ has commutant $M_2(\mathcal{A})$. In particular P lies in $M_2(\mathcal{A})$, since P commutes with $\pi(s)$. Similarly, since $U\pi(s)U^* = \pi(s^{-1})$, we find that $U_s m(f) U_s^* = m(\check{f})$ for any diagonal multiplication operator $m(f)$. Hence $U_s^*(J \otimes I)$ commutes with $\mathcal{A}$ and therefore lies in $M_2(\mathcal{A})$. So $U_s = M(J \otimes I)$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ and $Jf(x) = f(-x)$.

The coefficients of M can be calculated by comparing WUf and Wf when f is one of the test functions $\exp(-x^2/2)$ or $x\exp(-x^2/2)$. We find $a = d = (u/Ju + iv/Jv)/2$ and $b = c = (u/Ju - iv/Jv)/2$ where $u(x) = 2^{\frac{1}{2}ix + \frac{1}{4}}\, \Gamma(\frac{1}{2}ix + \frac{1}{4})/2\sqrt{2\pi}$ and $v(x) = -2^{\frac{1}{2}ix + \frac{3}{4}}\, \Gamma(\frac{1}{2}ix + \frac{3}{4})/2\sqrt{2\pi}$. Now $P_s = U_s Q_s U_s^* = MQ_sM^* = \begin{pmatrix} |a|^2 & a\bar{c} \\ c\bar{a} & |c|^2 \end{pmatrix}$. So $\Delta = QPQ/QP^{\perp}Q$ under W is multiplication by $|a|^2/|c|^2$ on the first copy of $L^2(\mathbf{R})$. Using the functional equation $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$, we obtain $|a|^2 = \frac{e^{\pi x}}{e^{\pi x} + e^{-\pi x}}$ Since $|a|^2 + |c|^2 = 1$, we find $|a|^2/|c|^2 = e^{2\pi x}$ so that Δ_s^{it} is multiplication by $\exp(2\pi i x)$ on the range of Q_s. Hence on $L^2(\mathbf{R}_+)$, $\Delta^{it} = \pi(\exp 2\pi t)$.

(2) For $f \in L^2(\mathbf{T})$ define $(V(t)f)(z) = (s_t z + c_t)^{-1} f(c_t z + s_t / s_t z + c_t)$ where $s_t = \sinh t$ and $c_t = \cosh t$. We must show that $\Delta^{it} = V(\pi t)$ on $L^2(I)$. Let H be the self-

adjoint generator of $V(t)$, so that $V(t) = \exp(iHt)$ and let C be the unitary $(Cf)(z) = -z^{-1}f(z^{-1})$. Furthermore let $T : QPH \to Q^\perp PH$ be the closed densely defined operator $T(Qf) = iQ^\perp f$ for $f \in PH$. It suffices to show that $T = C \cdot V(2\pi i) = C \cdot \exp(-2\pi H)$. The subspace $D = \{Qf : f = F|_{S^1}$ with F entire$\}$ is a core for T. Therefore it suffices to check that D lies in the domain of $V(-\pi i/2)$ and agrees with $C \cdot V(-i\pi/2)$ there. But for $-3\pi/4 < \operatorname{Im} w < \pi/4$, $g_w(z) = \chi_I(z)(s_w z + c_w)^{-1}F(c_w z + s_w/s_w z + c_w)$ defines an element of $L^2(I)$. The map $w \mapsto g_w$ is (weakly) holomorphic by Morera's theorem and satisfies $g_t = U(t)Qf$. So by the spectral theorem, Qf lies in the domain of $V(-i\pi/2)$ and $V(-i\pi/2)Qf = g_{-i\pi/2} = -iQCf$. Hence $T(Qf) = iQ^\perp f = i(CQC)f = CV(-i\pi/2)Qf$.

8. LOCAL FACTORISATION FOR FREE FERMI FIELD — Let I_1 and I_2 be subintervals of the circle whose closures are disjoint, so that they are separated on either side by open intervals, say J_1 and J_2. Let Q_1 and Q_2 be the projections given by the characteristic functions of I_1 and I_2 and let $Q = Q_1 + Q_2$. Let us start by observing that Q_1PQ_2 must be Hilbert-Schmidt (even trace-class). Indeed let f_1 and f_2 be smooth functions with disjoint support such that $f_k = 1$ on I_k. Then since $f_1f_2 = 0$, we have

$$Q_1PQ_2 = Q_1(f_1Pf_2)Q_2 = Q_1(f_1[P, f_2])Q_2,$$

which is Hilbert-Schmidt because $[P, f_2]$ already is. Thus on the CAR algebra of $I = I_1 \cup I_2$, ϕ_P induces the state ϕ_{QPQ}. But, modulo the Hilbert-Schmidt operators, QPQ agrees with $Q_1PQ_1 + Q_2PQ_2$. Since the CAR algebra of I acts cyclically, it follows that the restricted representation is unitarily equivalent to the graded tensor product of the restrictions of ϕ_P to the CAR algebras of I_1 and I_2 (cf [Buchholz; CMP 36], [Summers; CMP 86]).

9. THE TOPOLOGY ON SOBOLEV SPACE — For $f = \sum a_n z^n$ in $C^\infty \equiv C^\infty(S^1) \otimes \operatorname{End}(V)$ we define the Sobolev 1/2-norm $\|f\|_{1/2}$ by $\|f\|_{1/2}^2 = \sum_n (|n| + 1)|a_n|^2$ and the corresponding inner product by $(f, g)_{1/2}$. Sobolev space $L^2_{1/2} = L^2_{1/2}(S^1) \otimes \operatorname{End}(V)$ is the corresponding Hilbert space completion. Note that if $f \in C^\infty(S^1)$ with $f(z) = \sum a_n z^n$, then we have the crucial identity: $\|[P, m(f)]\|_2 = \sum |n||a_n|^2 = \|f\|_{1/2}^2 - \|f\|_2^2$ where P is the Hardy space projection and $m(f)$ is the multiplication operator of f.

For any finite set $A \subset S^1$ let $C^{\infty,A}$ denote the subspace of maps in C^∞ vanishing to infinite order on A. We claim that any map in C^∞ is a Sobolev limit of a sequence of uniformly bounded functions in $C^{\infty,A}$. In fact, by classical results of Shilov-Whitney on closed ideals (cf [Malgrange]), $C^{\infty,A}$ is dense in $C^{1,A}$, the ideal in C^1 of functions vanishing to first order on A (alternatively this may be checked locally by applying the Stone-Weierstrass theorem to first derivatives). So we need only approximate by functions in $C^{1,A}$. Let $a_n(z) = \operatorname{Im} z^k/n$ and $b_n(z) = f_{n-1/n}(z)/f_{n-1/n}(1)$ where $f_r(t) = \sigma_{k\geq 1} r^k \operatorname{Re} z^k/k \log k$. Then $a_n(1) = 0 = b_n'(1)$ and $a_n'(1) = 1 = b_n(1)$; moreover both a_n and b_n are uniformly bounded but tend to zero in $L^2_{1/2}$. Using a partition of unity, we obtain a similar sequence with support near a for each $a \in A$. The result follows.

Let G be any connected closed subgroup of $U(V)$ with Lie algebra $\mathfrak{g}$ in the skew-adjoint matrices. So $L\mathfrak{g}$ and LG lie in $C^\infty(S^1) \otimes \operatorname{End}(V)$. Let L^AG be the normal subgroup of LG of loops $g(z)$ satisfying $g(a) = 1$, $g^{(n)}(a) = 0$ for all $a \in A$. We show that L^AG is dense in LG for the Sobolev metric.

The exponential map $f \mapsto e^f$ takes skew-adjoint valued functions into unitary ones and is continuous at 0: for $\|[m(e^f), P]\|_2 \leq \|[m(f), P]\|_2$, since for example $e^fPe^{-f} - P = \int_0^1 e^{ft}([f, P])e^{-ft}\,dt$. Now let $f \in LG$. Using the b_n's we can find $X_n \in L\mathfrak{g}$ such that $\exp(X_n(a)) = f(a)$ $(a \in A)$ and $\|X_n\|_{1/2} \to 0$. Then $x_n = fe^{-X_n} \to f$ (in the Sobolev metric). Hence, taking $f_1 = x_n$ for n large, we have approximated f by f_1 with

$f_1(a) = 1$ for all $a \in A$. Using the a_n's we can find $Y_n \in L\mathfrak{g}$ with $\|Y_n\|_{1/2} \to 0$ and such that $Y_n(a) = 0$, Y_n of the form $h(z) \otimes Y$ near a, and $Y_n'(a) = f_1(a)^{-1}f_1'(a)$ for all $a \in A$. Then $y_n = f_1e^{-Y_n} \to f_1$. So if $f_2 = y_n$ for n large, we have approximated f_1 by f_2 with $f_2(a) = 1$, $f_2'(a) = 0$ for $a \in A$. Now we choose $Z \in L\mathfrak{g}$ such that $f_2 = e^Z$ in a neighbourhood of A with $Z(a) = Z'(a) = 0$ on A. We can find $Z_n \in L\mathfrak{g}$ vanishing to infinite order on A such atht $Z_n \to Z$ in the C^1 norm. Then $e^{Z_n} \to e^Z$ in the C^1 and hence the Sobolev metric. So $z_n = f_2e^{-Z}e^{Z_n} \to f_2$. Thus for large n, $f_3 = z_n$ is a good approximation to f_2 in L^AG.

10. THE QUARK MODEL — Let $H = L^2(S^1) \otimes V$ and let P be thr projection onto Hardy space $H^2(S^1) \otimes V$. The loop group $LU(V)$ may be identified with the unitary group of the algebra $C^\infty(S^1) \otimes \operatorname{End}(V)$. A acts on H by multiplication and commutes with P modulo the Hilbert-Schmidt operators by the well known Toeplitz property. So $LU(n)$ lies in $U_{res}(H)$. Diff S^1 also acts on H with image lying in $U_{res}(H)$ (see [PSegal], [Segal]). The corresponding projective representation of $LU(V) \rtimes \operatorname{Diff} S^1$ on the Fock space for P is called the basic (level one) representation. It is alraedy irreducible when restricted to LT, where T is a maximal torus of $U(V)$. If G is a closed subgroup of $U(V)$, we get a represntation of LG by restriction, in general with a level ℓ higher than one ($\ell = 1$ for $SU(V)$). All the irreducible representations of level ℓ will appear in this case.

✓ see §3 — current algebra

11. BOSONS AND THE FKS MODEL — The group LT is the direct product of the discrete group $\Lambda = \operatorname{Hom}(S^1, T)$ and the identity component $(LT)^0$. The fermionic representation can be realised in a completely different way using this observation. In fact, using the Mackey machine, any irreducible positive energy representation $(\pi, \mathcal{H})$ of LT is induced from a positive energy irreducible representation $(\pi_0, \mathcal{H}_0)$ of $(LT)^0$. The representation will be on the Hilbert space $\mathcal{H} \otimes \ell^2(\Lambda)$ and will involve the 2-cocycle on $\Lambda \times \Lambda$ determined by the restriction of π. Note that since $(LT)^0 = \exp L\mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of $\mathfrak{t}$, we are really talking about representations of an infinite Heisenberg group (i.e. the Weyl formulation of the canonical commutation relations). We recall some facts about the generalised Stone–von Neumann theorem.

winding numbers

Let V denote the real vector space $C^\infty(S^1, \mathbf{R})$ with non-degenerate symplectic form $S(f, g) = \int f'g$ (we insist $\int f = 0$ to get rid of zero modes). We consider projective representations $f \mapsto U(f) = \exp(i\phi(f))$ for which $U(f)U(g) = \exp(-iS(f, g))U(f + g)$ which have positive energy in an obvious sense (in fact, we need only demand the strong continuity of one parameter subgroups but this is technically harder). The generalised Stone–von Neumann theorem asserts that there is a unique such representation on bosonic Fock space. As observed in [Araki–Woods, JMP4], this representation is continuous for the norm defined by an appropriate inner product on V. To describe this inner product we must effectively polarise V into odd and even parts: a function f is even if $f(\bar{z}) = f(z)$ and odd if $f(\bar{z}) = -f(z)$.

We make the transition from the symplectic point of view to the Hilbert space point of view in the usual way. Indeed suppose that V is a real vector space with a non-degenerate symplectic form S and a given polarisation. So we assume that there for the weak topology defined by S, there is a total set of vectors p_i, q_i $(i \geq 1)$ satisfying $S(p_i, q_i) = \delta_{ij}$, $S(p_i, p_j) = 0$ and $S(q_i, q_j) = 0$. Let P and Q be the (algebraic) inner product spaces with orthonormal bases (p_i) and (q_i) respectively and identify P and Q by the map $J(p_i) = -q_i$, $J(q_i) = p_i$. Then $S((p, q), (p', q')) = (q, p') - (p, q')$. (In other words $V_0 = P \oplus JP$ makes V_0 into complex Hilbert space $P_{\mathbf{C}}$ with complex structure J and $S(\xi, \eta) = \operatorname{Im}(\xi, \eta)$.) Then defining $V(p) = U(Jp)$, we obtain the more standard form of the CCR, $U(f)V(g) = e^{-i(f,g)}V(g)U(f)$, for $f, g \in P$. As Araki and Woods show, the Fock space representations U and V of P are strongly continuous when P has the inner product norm and extend by continuity to the Hilbert space completion $\overline{P}$ of P. There is an obvious way of phrasing this in the symplectic picture.

To see what this means when we take V to be the subspace of real valued trigonometric polynomials f of mean 0 (i.e. $\int f = 0$) with symplectic form $S(f,g) = \int f dg$, take P and Q to be the subspaces with bases $p_n = \sqrt{2n}\cos nt$ and $q_n = \sqrt{2}\sin nt$ ($n \geq 1$) respectively. The Darboux conditions are readily checked and the map J becomes $J(\cos nt) = -n \sin nt$, $J(\sin nt) = n\cos nt$ for $n \geq 1$. So J may be identified with d/dt. The inner product on V becomes on the complexification $(\sum a_n z^n, \sum b_n z^n) = \sum |n| a_n \overline{b_n}$, which is equivalent to the Sobelev 1/2-norm on the mean zero functions. (We have had to remove the zero modes.)

We now deduce Haag duality for the free bosonic field on the circle by descent from the result for the free fermions. This was first done in the context of Minkowski space by Araki using the smeared quantum fields directly and was technically quite hard. We have already seen that for the fermionic representation π of $LU(1)$, the von Neumann algebras $\pi(L_I U(1))''$ and $\pi(L_{I^c} U(1))''$ are each other's quasi-commutants and that the modular group is geometric. The free bosonic field on S^1 (without charge or momentum operators) is described in the Weyl formulation by $LU(1)^0$, the loops of winding number zero, and the cocycle is unique up to rescaling. We restrict π to $LU(1)^0$ and take the subrepresentation π_0 on $\mathcal{H}_0$, say, generated by the vacuum vector. This gives the Fock (Stone–von Neumann) representation of the canonical commutation relations and we obtain the corresponding duality result by Takesaki devissage. (Note that $\mathcal{H}_0$ is invariant under Möbius transformations, since Ω is fixed by $SU(1,1)$.) To summarise: *if π_0 is the Fock cocycle representation of the additive group $\mathcal{B} = C^\infty(S^1, \mathbf{R})$ on $\mathcal{H}_0$, then $\pi_0(\mathcal{B}_I)'' = \pi_0(\mathcal{B}_{I^c})'$, where $\mathcal{B}_J$ denotes the subgroup of $\mathcal{B}$ supported in the interval J. The modular group and conjugation with respect to the vacuum vector are geometric.*

12. IRREDUCIBILITY OF LOOP GROUP SUBFACTORS. We prove directly that any inclusion $\pi(L_I G)'' \subset \pi(L_{I^c})'$ is irreducible whenever π is an irreducible positive energy representation of LG. This is equivalent to showing that $\pi(L_I G)' \cap \pi(L_{I^c} G)' = \mathbf{C}$. But this intersection if identical to the commutant of the subgroup of LG generated by $L_I G$ and $L_{I^c} G$, that is the normal subgroup of loops trivial to infinite order at the end points of I. So we have to show that π restricts to an irreducible representation of this normal subgroup. More generally, for any finite subset $A \subset S^1$, let $L^A G$ be the normal subgroup of loops trivial to infinite order at points of A and consider the restrictions of π to $L^A G$.

Let $H = L^2(S^1) \otimes V$ and let P be the projection onto Hardy space $H^2(S^1) \otimes V$. Any function $f \in C^\infty(S^1) \otimes End(V)$ acts by multiplication on H and satisfies $\|[P,f]\|_2 \leq (\int \|(f', f)\|)^{1/2} = \|f\|_{1/2}$, the Sobolev 1/2-norm. Thus, putting on $LU(V)$ the topology of almost everywhere convergence jointly with the metric $d(U,V) = \|U - V\|_{1/2}$, we get a continuous projective representation $LU(V) \to PU(H_P)$, the *basic representation*. All the level one representations of LG where G is simple are obtained by embedding G in some $U(V)$ and then composing the map $LG \to LU(V)$ with the basic representation. Since all higher level representations are abtained by decomposing tensor products of sufficeintly many level one representations, it follows that they are all continuous when we give LG the induced topology from $LU(V)$. (This does not depend on the particular $U(V)$ we choose.) We have already seen that $L^A G$ is dense in LG for this topology. So for any positive energy representation, $\pi(LG)$ is in the weak operator closure of $\pi(L^A G)$. Hence the von Neumann algebras they generate coincide as required.

In general we cannot realise all representations through complex fermions. However since any group is contained in a unitary group, we can always get fermionic representations at some possibly high level. (For example for E_8, the first level where this is possible is level 30.) For such a fermionic embedding, we get all the positive energy

representations of that level. On the other hand if π is the direct sum of all the level one representations, then all the level ℓ representations occur as summands of $\pi^{\otimes \ell}$. Now the topology induced on LG by the homomorphism $LG \to PU(\mathcal{H})$ is clearly the same as that induced by the homomorphism $LG \to PU(\mathcal{H}^{\otimes \ell})$, since the map $PU(\mathcal{H}) \to PU(\mathcal{H}^{\otimes \ell})$, $g \to g^{\otimes \ell}$ is a homeomorphism onto its image. So from the topology on $LU(n)$ will give the right topology on LG for any $G \subset U(n)$.

The topology on LG can also be determined using bosons, since, when G is simply connected, LG is generated by copies of $(LT)^0$. The restriction of any positive energy representation π of LG to $L\mathbf{T}^0 = \exp(iV)$ yields the unique Heisenberg representation (with a multiplicity). In this representation $L^A \mathbf{T}^0$ is dense in $L\mathbf{T}^0$. Hence $\pi(L^A \mathbf{T}^0)$ is dense in $\pi(L\mathbf{T}^0)$. It follows that $L^A G$ is dense in LG in the topology defined by any positive energy representation.

A third argument can be given based on the Mackey machine for $LSU(2)$. We have a gauge group with a compact quotient group, a product of $SU(2)$'s and can use the fact that, if $SU(2)$ acts ergodically on an operator algebra, the algebra must be type I. This approach has technical complications and to be completed needs some facts about vertex operators.

Note that these arguments are not applicable to Diff S^1. The topology on Diff S^1 induced by the weak operator topology on $U(L^2(S^1))$ is stronger than the uniform topology. The reason for this is the need to correct by the derivative of the diffeomorphism even when defining its action on constant functions.

13. HAAG DUALITY IN THE VACUUM SECTOR — We have already proved Haag duality in the vacuum sector for free fermions. Consider the basic representation π of $LU(n)$ obtained through free fermions on $L^2(S^1) \otimes \mathbf{C}^n$ for $n \geq 1$. The next two results give us Haag duality for $LU(n)$ and for bosons, in the LT formulation.

LEMMA. — $\pi(L_I U(n))'' = \pi_P(\mathrm{Cliff}(L^2(I) \otimes \mathbf{C}^n))''$. *Denote this algebra by M. If U_t is the modular group of M with respect to ξ_P, then $\xi_P^{\otimes n}$ is (up to scalar multiples) the unique vector fixed by $U_t^{\otimes n}$. So the modular group of $M^{\otimes n}$ for the vector $\xi_P^{\otimes n}$ is ergodic. In particular M is isomorphic to the hyperfinite type III_1 factor.*

PROOF. Note that $L_I U(n)$ quasi-commutes with fermions supported in I^c, so the left hand side is contained in the right hand side. The modular group of the right hand side with respect to the vacuum vector ξ_P is the Möbius flow and leaves the left hand side invariant. On the other hand by the Reeh–Schlieder theorem ξ_P is cyclic for the left hand side. So by Takesaki's theorem we have equality.

The action of $SU(1,1)$ on $\mathcal{H}_P$ is the direct sum of one copy of the trivial representation (on $\mathbf{C}\xi_P$) and various holomorphic discrete series representations. So the representation $\mathcal{H}_P$ of U_t is a sum of the trivial representation and an infinite number of copies of the regular representation. Tensoring any representation σ with the regular representation gives $\dim\sigma$ copies of the regular representation. So U_t and $U_t^{\otimes n}$ are isomorphic as representations of $\mathbf{R}$. In particular $\xi_P^{\otimes n}$ is the unique fixed ray. (This should also follow from simple spectral theory, for operators $A \otimes I + I \otimes B$ where A and B are self-adjoint.) From section 4.4, we deduce the statements about ergodicity and type. Hyperfiniteness is clear since the CAR algebra is a UHF algebra.

LEMMA. — *Haag duality holds for LT^0 (without zero modes) and the local algebras are isomorphic to the hyperfinite III_1 factor and the modular group is geometric.*

PROOF. We have already seen that Haag duality holds for the fermionic representation π of $LU(V)$. If T is a maximal torsu in $U(V)$, the bosonic representation of LT^0 can be obtained by restricting π. Applying Takesaki devissage, we obtain Haag duality for the vacuum bosonic representation along with the other assertions (by ergodicity of the modular group).

LEMMA. — *Haag duality holds in the vacuum sector for simply laced groups at level one and the modular group is geometric. The local algebras are isomorphic to the hyperfinite III_1 factor.*

PROOF. For $SU(n)$ or $SO(2n)$ this would follow immediately by devissage from complex or real fermions. To include the exceptional cases, E_6, E_7 and E_8, we use bosons. Let T be a maximal torus of G. The vacuum representation of LG on $\mathcal{H}$ remains irreducible on LT. We claim that $\boxed{\pi(L_IT)'' = \pi(L_IG)''}$. We first observe that T acts on $\pi(L_IT)''$ with fixed point algebra $\pi(L_IT^0)''$ and leaving fixed the vacuum state: for $\mathcal{H}$ is induced so isomorphic to Fock space tensored with $\ell^2(\Lambda)$ (where $\Lambda = \mathrm{Hom}(S^1,T)$) with the obvious action of $L_IT^0 \times T$. The modular group for $\pi(L_IT)''$ must restrict to the modular group on the fixed point algebra of T by KMS uniqueness, and hence is geometric there. On the other hand we can show directly that $\pi(L_IT)'' = \pi(L_{I^c}T)'$: for the action of T shows that $\pi(L_IT)''$ is a cocycle crossed product of $\pi(L_IT^0)''$ by Λ concretely realised on $\mathcal{H}$ so the commutant is easy to compute. (Explicitly, we find loops $u_\alpha \in L_IT$ in the same homotopy class as $\alpha \in \Lambda \subset LT$. So $L_IT = \cup u_\alpha L_IT^0$ with the $\pi(u_\alpha)$'s eigenvectors for $\hat{\Lambda} = T/\Lambda^\perp$. Evidently $u_\alpha u_\beta = c(\alpha,\beta)u_{\alpha\beta}$ for some cocycle c in $\pi(L_IT^0)''$ and the u_α's define a cocycle coaction.) But then $\pi(L_IG)'' = \pi(L_{I^c}G)' = \pi(L_IT)''$ by conformal sandwiching. So the modular group of $\pi(L_IG)''(\cong \pi(L_IT)'')$ restricts to the geometric group on $\pi(L_IT^0)$. Conjugating by constant loops in G, we see that this is true for any maximal torus. But L_IG is generated by the subgroups L_IT^0 as T varies, and hence the modular group must be geometric on the whole of $\pi(L_IG)''$. The modular group is ergodic, so factoriality and type follow. Hyperfiniteness follows because the group T acts on the local algebra with hyperfinite fixed point algebra (any cocycle crossed product of a hyperfinite algebra by a discrete abelian group is injective: note the action can be untwisted for it is equivalent to the action by conjugation of Λ, so we can use Connes' result on crossed products by abelian groups. This untwisting is predicted by Theorem 4.3.3 in [Sutherland]. See also [Sutherland], Theorem 6.2 or [Popa–Wassermann]).

PROPOSITION. — *Suppose that Haag duality holds for two vacuum representations π_1 and π_2 of LG and that the modular groups for L_IG are geometric and ergodic, with hyperfinite local algebras. Then the same is true for the vacuum subrepresentation π of $\pi_1 \otimes \pi_2$. Indeed there is a unique conditional expectation of $\pi_1(L_IG)'' \otimes \pi_2(L_IG)''$ onto $\pi(L_IG)''$ commuting with the tensor product of the modular groups and this tensor product is ergodic. So $\pi(L_IG)''$ is a hyperfinite type III_1 factor. (The result extends immediately to any finite number $\pi_1, \dots, \pi_n$ of vacuum representations.)*

PROOF. Let $M_i = \pi_i(L_G)''$ and $M = \pi(L_IG)$. Now apply Takesaki devissage to the inclusion $M \subset M_1 \otimes M_2$ and the vector $\xi_1 \otimes \xi_2$. (The ξ_i's are vacuum vectors.) vacuum?

THEOREM. — *Haag duality holds for representations of a loop group of a simple connected Lie group at any level. The von Neumann algebra of L_IG has a geometric and ergodic modular group and is isomorphic to the hyperfinite type III_1 factor.*

PROOF. In view of the previous lemma it suffices to prove the result at level one since the level ℓ vacuum representation is obtained by picking out the obvious summand of the ℓth tensor power of the one at level one. At level one this can be done in variety of ways. For example for $SU(n)$, the level one representations arise by restricting the fermionic representation π of $LU(n)$. They all occur with infinite multiplicity and the multiplicity spaces furnish positive energy representations of a commuting copy of $LU(1)$, where $U(1)$ here is the centre of $U(n)$. (In fact π breaks up as a sum of n representations of $LSU(n) \times LU(1)$. This can be seen readily by noting that $LU(n)/LSU(n)LU(1)$ is a cyclic group of order n and then using Mackey machine arguments.) The result then follows by Takesaki devissage. For the other family of simply laced classical groups $Spin(2n)$,

a similar argument would apply if we had studied real (or Majorana) fermions. But we have not, so instead we use bosons. Recall that if G is simply laced with maximal torus T, then the level one vacuum representation π of LG restricts to an irreducible representation of LT. This representation is of the type before, so $\pi(L_IT)'' = \pi(L_{I^c}T)'$. On the other hand $\pi(L_IG)''$ and $\pi(L_{I^c}T)'$ are sandwiched between these two algebras and hence are equal and have all the claimed properties. It remains to handle the groups $B_n = \mathrm{Spin}(2n+1)$, $C_n = U(n, \mathbb{H})$ and the exceptional groups G_2 and F_4. The latter are dealt with by a conformal inclusion $G_2 \times F_4 \subset E_8$ all at level one, which simultaneously establishes the local equivalence for these exceptional groups at level one. Similarly the cases B_n and C_n are dealt with using the conformal inclusions $(B_m)_1 \times (B_n)_1 \subset D_{m+n+1}$ and $(C_n)_1 \times SU(2)_m \subset D_{2m}$.

We now extend this result to vacuum representations in the discrete series of Diff S^1.

THEOREM. — *Let π be a vacuum discrete series representation of Diff S_1. Then $\pi(\mathrm{Diff}_I S^1)'' = \pi(\mathrm{Diff}_{I^c} S^1)'$. Moreover the modular group of this algebra (with respect to the vacuum vector) is geometric and ergodic. Moreover the algebra is isomorphic to the hyperfinite type III_1 factor.* $c<1$

PROOF. We may use any of the GKO constructions (there are three families that give the whole discrete series). For example we may take the vacuum representation π of $G = SU(2) \times SU(2)$ at level $(n,1)$ which contains $H = SU(2)$ at level $n+1$ through the diagonal embedding. The GKO representation of Diff S^1 is defined by $\pi_{\mathrm{GKO}}(f) = \pi_G(f)\pi_H(f)^*$ where π_G and π_H are the Segal–Sugawara representations of Diff S^1 for G and H respectively. (See below for a complete explanation.) In particular, this formula shows that $\pi_{\mathrm{GKO}}(\mathrm{Diff}_I S^1)'' \subset \pi(L_IG)''$ and that the subalgebra is invariant under the modular group of $\pi(L_IG)''$. This modular group is the tensor product of two ergodic and geometric modular groups, so is in turn ergodic and geometric. The restriction to the subalgebra is clearly geometric in the sense that it is implemented by π_{GKO}. So the result follows by Takesaki devissage. (Note that the vacuum vector for LG is the tensor product of the vacuum vectors of LH and Diff S^1 uder the GKO decomposition.)

14. LOCAL AUTOMORPHISMS AND THE CENTRE — The centre Z of G acts by outer automorphisms on LG: indeed take conjugation by any path γ_z from g to zg with smooth derivatives for $z \in Z$. As explained in [Segal], conjugation $\pi \circ \mathrm{Ad}\,\gamma_z$ gives a new positive energy representation π_z from a given one π. The equivalence class of π_z depends only on z and not the particular path chosen. In other words the centre of G acts naturally on the (equivalence classes of) positive energy representations of LG. Now given an interval I we can always arrange for γ_z to stay stationary on I. Consequently the representations π_z and π are manifestly equivalent on L_IG. When G is simply laced, the centre acts simply transitively on the level one representations, so they are all locally equivalent. Moreover all these sectors satisfy Haag duality, because $\mathrm{Ad}\,\gamma_z(L_IG) = L_IG$.

The above arguments can be refined to show the failure of Haag duality in the vacuum sector at level one for disjoint intervals. In fact we subdivide the circle into four contiguous intervals I_1, I_2, I_3 and I_4. Take a loop $\gamma_1 \in LT$ equal to 1 on I_1 and z_1 on I_3 and a loop $\gamma_2 \in LT$ equal to 1 on I_2 and z_2 on I_4, where $z_1, z_2 \in Z$. Let $I = I_1 \cup I_3$ so that $I^c = I_2 \cup I_4$. Then $\pi(\gamma_1)$ lies in $\pi(L_IG)'$ and $\pi(\gamma_2)$ lies in $\pi(L_{I^c}G)'$, but these elements do not commute as one can verify directly from the cocycle formula in [PSegal].

15. LOCAL EQUIVALENCE AND TYPE —

THEOREM. *Two irreducible positive energy representations of a given level restrict to unitarily equivalent representations of L_IG. The associated von Neumann algebras are type III_1 hyperfinite.*

PROOF. We start by showing that all the level one representations are locally equivalent.

For simply laced groups, we proved this above and for C_n and the exceptional groups G_2 and F_4 this was already noted using the conformal inclusions $(C_n)_1 \times SU(2)_m \subset D_{2m}$ and $G_2 \times F_4 \subset E_8$.

With the level one result at our disposal, we see that if $\pi_1, \ldots, \pi_\ell$ are any level one irreducible representations, then $\pi_1 \otimes \cdots \otimes \pi_\ell$ is a direct sum of level ℓ representations of LG and evry representation can be obtained as such a summand. On $L_I G$ it is unitarily equivalent to $\pi_0^{\otimes \ell}$, where π_0 is the vacuum representation. On the other hand $\pi_0^{\otimes \ell}(L_I G)$ is a hyperfinite type III_1 factor. Hence all its subrepresentations are unitary equivalent. Hence all the level ℓ representations are unitarily equivalent on $L_I G$.

PROPOSITION. — *If $f \in \mathrm{Diff}_I S^1$ and π is an irreducible positive energy representation of LG extended to $LG \rtimes \mathrm{Diff}\, S^1$, then $\pi(f) \in \pi(L_I G)''$. The positive energy representations of a given level are all unitarily equivalent on $L_I G \rtimes \mathrm{Diff}_I S^1$. The Segal-Sugawara representations of the diffeomorphism group at a fixed level are all unitarily equivalent factor representations of $\mathrm{Diff}_I S^1$.*

PROOF. Let π_0 be the vacuum representation at the same level as π and let $U : \mathcal{H}_0 \to \mathcal{H}$ be a unitary intertwiner for $L_I G$ between π_0 and π. Thus $U\pi_0(\gamma)U^* = \pi(\gamma)$ for $\gamma \in L_I G$. Now $\pi_0(f)$ lies in $\pi_0(L_{I^c}G)'$ which is the same as $\pi_0(L_I G)''$ by Haag duality. So $T = U\pi_0(f)U^*$ lies in $\pi(L_I G)''$. Now $T\pi(\gamma)T^* = U\pi_0(\gamma \circ f)U^* = \pi(\gamma \circ f) = \pi(f)\pi(\gamma)\pi(f)^*$, so that $T^*\pi(f)$ lies in $\pi(L_I G)'$. But T lies in $\pi(L_I G)''$ and $\pi(f)$ lies in $\pi(L_{I^c}G)'$, so $T^*\pi(f)$ lies in $\pi(L_{I^c}G)' \cap \pi(L_I G)' = \mathbb{C}$ by irreducibility. Hence $\pi(f)$ is a multiple of T and therefore in $\pi(L_I G)''$. Since $\pi(f)$ and $U\pi_0(f)U^*$ are proportional, the second assertion follows. Finally we know the vacuum representation restricts to an (injective) factor representation of $\mathrm{Diff}_I S^1$ so the last assertion follows from the second.

COROLLARY. *If π is a positive energy representation of LG then there is a unique projective representation of $\mathrm{Diff}_I S^1$ in $\pi(L_I G)''$ such that $\mathrm{Ad}\,\pi(f) \cdot \pi(\gamma) = \pi(\gamma \circ f)$. Hence if U is a unitary intertwiner between two different representations π and σ of $L_I G$, it automatically intertwines the associated representation of $\mathrm{Diff}_I S^1$.*

PROOF. Existence follows from the proposition and uniqueness follows from the fact that $\pi(L_I G)''$ is a factor. The uniqueness result implies that π and $\mathrm{Ad}\,U^*\sigma$ must agree.

LEMMA. — *The discrete series representations of $\mathrm{Diff}\, S^1$ for a fixed central charge are all unitarily equivalent (injective) factor representations.*

PROOF. Suppose that the discrete series is associated with a an inclusion $LH \subset LG$ and π is a positive energy irreducible representation of $L_I G$ of level one. From the above corollary, $\mathrm{Diff}_I S^1$ has unique projective representations π_1 and π_2 in $\pi(L_I G)''$ and $\pi(L_H)''$ respectively, compatible with its reparametrisation actions on $L_I G$ and $L_I H$. Now $\mathrm{Ad}\,\pi_1$ and $\mathrm{Ad}\,\pi_2$ agree on $L_I H$ and commute by the argument of [GKO] (cf section 3). Thus the GKO representation of $\mathrm{Diff}_I/, S^1$ given by $\pi_3(f) = \pi_1(f)\pi_2(f)^*$ lies in $\pi(L_I G)''$ and $\mathrm{Ad}(\pi_1(g)) \cdot \pi_3(f) = \pi_3(gfg^{-1})$. We shall write π_{GKO} for π_3: it is a sum of all the GKO representations coming from π, each with infinite multiplicity.

Taking π to be the vacuum representation of $L_I G$, we deduce by Takesaki devissage that π_{GKO} is a type III_1 hyperfinite factor representation. If σ is any other representation of level one, then there is a unitary U intertwining π and σ. By the corollary, U intertwines π_1 and π_2 with σ_1 and σ_2. Hence, by definition of π_{GKO} and σ_{GKO}, U must also intertwine π_{GKO} and σ_{GKO}. The result follows.

16. LOCAL FACTORISATION AND THE DICK TRICK.

THEOREM. — *Let π be a positive energy representation of LG and let I_1 and I_2 be disjoint intervals on the circle with $I = I_1 \cup I_2$. Then the restriction of π to $L_I G$ is unitarily equivalent the tensor product of the restrictions to $L_{I_1}G$ and $L_{I_2}G$. An analogous result holds for $\mathrm{Diff}\, S^1$.*

PROOF We have already seen that the analogous result is true for free fermions and $LU(n)$ in the fermionic representation. Since the result states that $\pi|_{L_I G} \cong \pi|_{L_{I_1} G} \otimes \pi|_{L_{I_2} G}$, we see that it passes immediately to subgroups $H < G$ and tensor products using the local equivalence of representations of a fixed level. So in particular it holds for $LSU(n)$ at any level and follows for any group once we know the result at level one. Now at level one when the group is simply laced, we have already observed that $\pi(L_J G)'' = \pi(L_J T)''$ whenever T is a maximal torus in G and J is a connected interval. This result extends immediately to finite unions of intervals. Tensor products and the result for the fermionic representation of $LU(1)$ show that the theorem is valid for LT^0. To deduce it for LT, we note that $L_I T$ is generated by $L_I T^0$ and elements u_α, v_α $(\alpha \in \Lambda)$ in the same homotopy class as α. Now we can implement the action of T on $L_{I_j} T$ by loops concentrated in a small neighbourhood of $\overline{I}_j$, so on $u_\alpha v_\beta$ are (α, β) eigenvectors of $T \times T$. This shows that the algebra $\pi(L_I T)''$ is the twisted crossed product of $\pi(L_I T^0)''$ by $\Lambda \times \Lambda$ and hence that factorisation holds also for LT (the von Neumann algebra generated by the $\pi(L_{I_j})''$ is isomorphic to their tensor product). Thus $\pi|_{L_I T} \cong \pi|_{L_{I_1} T} \otimes \pi|_{L_{I_2} T}$. Let U intertwine these two representations. We claim it also intertwines the corresponding subgroups of LG. In if $t_k \in L_{I_k} T$, then $U\pi(t_1)\pi(t_2)U^* = \pi(t_1) \otimes \pi(t_2)$, so by linearity we find that $U(a_1 a_2)U^* = a_1 \otimes a_2$ for $a_k \in \pi(L_{I_k} T)''$. Since $\pi(L_{I_k} G) \subset \pi(L_{I_k} T)''$, we get the result for LG. This proves the result at level one for simply laced groups and as above it follows for the other compact groups using conformal inclusions. The observation on tensor products implies the general assertion.

It remains to discuss $\mathrm{Diff}\, S^1$. Indeed since $\pi_{\mathrm{GKO}}(\mathrm{Diff}_I S^1)'' \subset \pi(L_I G)''$, the assertion follows immediately from the identity $Ua_1 a_2 U^* = a_1 \otimes a_2$ for $a_k \in \pi(L_{I_k} G)''$.

LEMMA. — *Suppose that M is a von Neumann algebra that can be obtained as the weak closure of an increasing union of algebras M_n. Suppose moreover that we can find a type I factor lying between M_n and M_{n+1} for every n. Then M is hyperfinite.*

PROOF. Obvious.

LEMMA. — *If $I \subset J$ are open interval on S^1, then there is a type I factor lying between $\pi(L_I G)''$ and $\pi(L_J G)''$. Moreover $\bigvee \pi(L_{I_n} G)'' = \pi(L_J G)''$ if the intervals I_n increase to J. Hence the local algebras are hyperfinite. A similar result holds for the diffeomorphism group.* split

PROOF. If H_1 and H_2 are groups with a representation π of $H_1 \times H_2$ such that $\pi|_{H_1 \times H_2} \cong \pi_{H_1} \times \pi_{H_2}$, then there is a type I factor between $\pi(H_1)''$ and $\pi(H_2)'$.

We mention a proof of factorisation based on a simplified version of [BW] and [BD'AF]. Assume we have a positive energy representation $U(z) = z^{L_0}$ of the circle group on $\mathcal{H}$ with vacuum vector Ω and that x and y lie in disjoint local algebras $M(I_1)$ and $M(I_2)$, so that $[x, U(z)yU(z)^{-1}] = 0$ for z near 1, say on an arc I with end points $a_\pm$ with $\overline{a}_+ = a_-$. Define $f_+(z) = (xz^{L_0}y\Omega, \Omega)$ for $|z| \leq 1$ and $f_-(z) = (yz^{-L_0}x\Omega, \Omega)$ for $|z| \geq 1$. The commutativity condition on x and y shows that f_+ and f_- agree on I so jointly define a holomorphic function f on $\mathbb{C}\backslash I^c$. Let $g(z) = \exp(-\alpha(z/a - 1)^{-1/2} - \overline{\alpha}(z/\overline{a} - 1)^{-1/2})$ for z in $\mathbb{C}\backslash I^c \cup (-\infty, -1]$, where $\alpha = \exp(-i\pi/4)$. This holomorphic function blows up at $a_\pm$; however in the closed sector S bounded by the radii through $a_\pm$ it satisfies $|g(z)| \leq 1$ and is continuous. Let Γ be any simple closed contour in S, coinciding with the radii near $a_\pm$ and winding round 1 once. If D is the domain enclosed by Γ, fg is holomorphic on D and continuous on $\overline{D}$. By Cauchy's theorem $2\pi i f(1)g(1) = \int_\Gamma g(z)f(z)(z-1)^{-1}dz$. Let Γ_+, Γ_- be the parts of the contour inside and outside the unit disc. Because $|g(ra_\pm)| \sim \exp(-(2|r-1|)^{-1/2})$ and there is an asymptotic estimate ([Kac-Wakimoto]) $\mathrm{Tr}(|(ra_\pm)^{L_0}|) = \mathrm{Tr}(|r|^{L_0}) \sim \exp(-C/\log r)$ with $C > 0$ as $r \uparrow 1$, we

see that
$$A_\pm = \frac{1}{2\pi i g(1)} \int_{\Gamma_\pm} g(z) f(z) z^{\pm L_0} (z-1)^{-1}\, dz$$
are trace class operators such that
$$(xy\Omega, \Omega) = f(1) = (xA_+ y\Omega, \Omega) + (yA_- x\Omega, \Omega)$$
for $x \in M(I_1)$ and $y \in M(I_2)$. Since A_+ and A_- are trace class, the right hand side extends to a normal form on $M(I_1) \odot M(I_2)$ which is a state ω in view of the form of the left hand side. The representation π_ω of $M(I_1) \odot M(I_2)$ is faithful (since the algebra is a factor) and may be canonically identified with the obvious representation on the closure of $M(I_1)M(I_2)\Omega$. By the Reeh–Schlieder theorem this is dense and thus π_ω gives an isomorphism of $M(I_1) \otimes M(I_2)$ onto the von Neumann algebra generated by $M(I_1)$ and $M(I_2)$. Because everything is type III, this isomorphism can be implemented by a unitary.

17. Localised endomorphisms and braid group representations. — We prove that the subfactors defined by LG or $\mathrm{Diff}\, S^1$ come equipped with localised endomorphisms and braiding, just like the Jones–Wenzl subfactors. Given the results on local equivalence, a postulate in algebraic QFT, we can argue as in [DHR], with slight differences due to our more concrete setting. We shall use $\mathcal{G}$ to stand for either LG or $\mathrm{Diff}\, S^1$ and $\mathcal{G}_I$ will be the subgroup supported in I.

Let $(\pi_0, \mathcal{H}_0)$ be the vacuum sector and $(\pi_1, \mathcal{H}_1)$ an arbitrary sector. Let I be an interval and with subinterval $I_1 \subset\subset I$. Set $J = I_1^c$. Let $U, V : \mathcal{H}_0 \to \mathcal{H}_1$ be unitary intertwiners for $\mathcal{G}_I$ and $\mathcal{G}_J$ respectively. Let $N = \pi_1(\mathcal{G}_I)''$ and $M = \pi_1(\mathcal{G}_{I^c})'$, so that $N \subset M$. Then if $M(I) = \pi_0(\mathcal{G}_I)''$ we have $M(I) = V^* M_1 V = U^* N_1 U$, the latter by Haag duality. Hence the canonical inclusion $N \subset M$ is isomorphic to the inclusion $V^* U M(I) U^* V \subset M(I)$ and $\rho(x) = V^* U x U^* V$ defines a endomorphism of $M(I)$ onto the subfactor. This endomorphism is localised with support in I_1 in the following sense: if x is has support either disjoint from I_1 or containing I_1, then $\rho(x)$ has the same support. When the support is disjoint, $\rho(x) = x$. (Note too that $\mathrm{Ad}(UV^*)$ defines an endomorphism of M onto N.) Clearly if ρ_1 and ρ_2 are localised endomorphisms of $M(I)$ with disjoint supports, then the support preserving properties of ρ_1 and ρ_2 immediately imply $\rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1$. It should be noted that in general if $J \subset\subset I$, we do not have $M(I) \cap M(J)' = M(I \backslash \overline{J})$: this is true however if J and I have a common endpoint, by Haag duality.

Let ρ be an endomorphism supported in $I_1 \subset\subset I$ and let $I \backslash \overline{I_1} = I_2 \cup I_3$. If T is a diffeomorphism in $\mathrm{Diff}_I\, S^1$, then $\rho_T(x) = T^* \rho(TXT^*) T$ is also a localised endomorphism. (No extra generality is obtained by allowing diffeomorphisms that fix the endpoints of I.) Note that $\rho_T = \mathrm{Ad}(U) \cdot \rho$ where $U = T^* \rho(T)$. Choose T so that ρ and ρ_T have disjoint supports. From $\rho \circ \rho_T = \rho_T \circ \rho$, we see that $g = U^* \rho(U)$ commutes with $\rho^2(M)$.

We claim (a)the element g is supported in I_1; (b) depends only on the relative position of I_1 and its image under T, changing to g^{-1} if this is reversed; and (c) satisfies the braid relation $g\rho(g)g = \rho(g)g\rho(g)$. It follows that $\pi(\sigma_i) = \rho^{i-1}(g)$ $(i \geq 1)$ defines a representation of B_∞ in M, with $\pi(\sigma_i)$ lying in the relative commutant of $\rho^{i+1}(M)$.

To see this, suppose ρ_T is supported in I_3. Then U is supported in $I_1 \cup I_3$ and $\rho(x) = U^* x U$ if x has support in I_1: for U lies in $M(I) \cap M(I_2)' = M(I_1 \cup I_3)$, since both ρ_T and ρ fix $M(I_2)$. But then $\rho(U)$ and hence $g = U^* \rho(U)$ has support in $I_1 \cap I_3$. If now x is supported in I_3, so too is $\rho_T(x) = UxU^*$. Hence UxU^* is fixed by ρ, forcing g to commute with all such x thus proving (a). If S is another diffeomorphism with ρ_S supported in I_3 and $V = S^* \rho(S)$, then both $\rho(x) = U^* x U = V^* x V$ for $x \in M(I_1)$.

Since UV^* is supported in $I_1 \cup I_3$, it must lie in $M(I_3)$ and therefore is fixed by ρ. Thus $U^* \rho(U) = V^* \rho(V)$, proving the first part of (b). The second part follows by applying the identity $\rho \cdot \rho_T = \rho_T \cdot \rho$ to T. To establish (c), we note that from the identity $g\rho(g) = V^* \rho^2(V)$, we obtain $\rho(g) g \rho(g) = \rho(g) V^* \rho^2(V)$ and $g\rho(g)g = V^* \rho^2(V) g = V^* g \rho^2(V)$, since $g \in \rho^2(M)'$. So the relation is equivalent to $V\rho(g)V^* = g$. This holds because $V\rho(g)V^* = \rho_T(g)$ and g and ρ_T have disjoint supports.

18. Examples of finite index through conformal and GKO inclusions. — There are two equivalent ways that an irreducible inclusion $N \subset M$ can have finite index in a computable way. The first is that there should be normal conditional expectations from M onto N and from N' onto M', both necessarily unique ([Connes]). (This condition is immediately equivalent to the condition of [Kosaki], by Haagerup's results on operator valued weights.) The second condition is that there should be a normal conditional expectation E of N onto M which satisfies an Pimsner–Popa inequality $E(x) \geq \lambda x$ for all $x \geq 0$ in N for some fixed $\lambda > 0$. The *index* is the reciprocal of the largest possible λ. In this case there is automatically a normal conditional expectation of N' onto M' satisfying the same inequality. The first condition is very natural for the loop group or diffeomorphism group subfactors, since it is completely symmetric. We have not been able to verify it directly for these inclusions: the usual 'mass gap' techniques fail in CFT. On the other hand, it is easy to see (by Haag duality) that the Möbius flow is a modular group for a weight on $\pi(L_I G)'$. Finite index amounts to this weight being a state, by Takesaki's theorem.

We now outline a method for establishing finite index using conformal inclusions based on the second 'Pimsner–Popa' criterion for finite index. Let $LH \subset LG$ be a conformal inclusion and suppose that the vacuum representation of LG restricts to representations of LH that all satisfy Haag duality or have finite index. Then the inclusion $\pi(L_I H)'' \subset \pi(L_I G)''$ has finite index and is independent of the given sector of π. If futhermore Haag duality holds for π (and G) and σ is an irreducible representation of LH appearing in $\pi|_{LH}$, then the inclusion $\sigma(L_I H)'' \subset \sigma(L_{I^c} H)'$ has finite index. Similar results hold for GKO inclusions $\mathrm{Diff}\, S^1 \times LH \subset LG$.

To see this set $M = \pi(L_I G)''$, $N = \pi(L_I H)''$ and $M_1 = \pi(L_{I^c} H)'$, so that $N \subset M \subset M_1$.

(1) By local equivalence, the inclusion $N \subset M$ is independent of π.

$M \subset M_1 : (L_I G)'' = (L_{I^c} G)' \subset (L_{I^c} H)'$

$M' \subset N' : (L_I G)' \subset (L_I H)'$

(2) If π satisfies Haag duality, the inclusion $M \subset M_1$ is isomorphic to the commutants of the inclusion $N \subset M$ (replacing I^c by I). So $N \subset M$ has finite index iff $M \subset M_1$ has finite index and they will have equal index.

(3) The inclusion $N \subset M$ (and hence $M \subset M_1$) is irreducible. When π is the vacuum representation, $N' \cap M_1 = \pi(LH)'$. We can do the computation $N' \cap M$ in the vacuum representation from (1). Note that $N' \cap M_1 = \pi(L_I H)' \cap \pi(L_{I^c} H)' = \pi(LH)'$ by density of $L_I H \cdot L_{I^c} H$ in LH. Since the vacuum vector is cyclic for $M' = \pi(L_{I^c} G)''$ (by the Reeh–Schlieder theorem), no non-trivial projection in $\pi(LH)'$ can lie in M and hence $N' \cap M = \mathbf{C}$.

(3) $N' \cap M_1 = \pi(LH)'$ = finite by §3. (conformal)

*) (4) We argue that for the vacuum representation, the inclusion $N \subset M_1$ is of finite index (in the sense of Kosaki). Let $N \subset M \subset M_1$ be inclusions with $N \subset M_1$ locally trivial, i.e. such that $N' \cap M_1$ is Abelian with minimal projections $p_1, \ldots, p_n$ with $p_j M_1 p_j = N p_j$. Set $E_1(x) = \sum p_j x p_j$ and let $u(\zeta) = \sum \zeta^i p_i$ for $\zeta \in \mu_n$ (the nth roots of unity). Since E_1 can be obtained by averaging over $\mathrm{Ad}\, u$, we see that E_1 is a conditional expectation onto the relative commutant of $u(\mu_n)$, i.e. onto $N \otimes (\oplus \mathbf{C} p_j)$, and that $E_1(x) \geq x/n$ for $x \geq 0$.. The trace on the second factor giving equal weight to the p_j's defines a conditional expectation E_2 of $N \otimes (\oplus \mathbf{C} p_j)$ onto N satisfying $E_2(x) \geq x/n$, so $E = E_2 \circ E_1$ gives a faithful normal conditional expectation of M_1 (and hence M) onto

only for conformal inclusion, otherwise $\mathrm{Diff} \subset N' \cap M_1$

rather discuss

$N = \pi(L_I H \times \mathrm{Diff}_I)'' \subset M \subset$

$\subset M_1 = \pi(L_{I^c} H \times \mathrm{Diff}_{I^c})'$

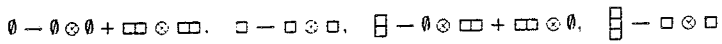
N satisfying $E(x) \geq x/n^2$ for $x \geq 0$. If in addition N is irreducible in M, we know that is at most one normal conditional expectation of M onto N. Thus the normal conditional expectation E of M onto N satisfies the Pimsner–Popa inequality. So $N \subset M$ has finite index. (A slightly more general argument is needed to handle the case when $\pi|_{LH}$ is not multiplicity free or where the inclusion is locally of finite index rather than just locally of index one. No new ideas are involved.)

The result follows by combining observations (1), (2), (3) and (4).

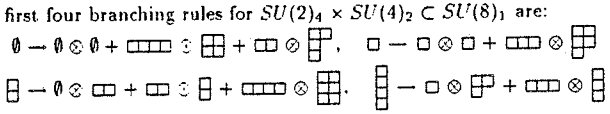
Note that M_1 is just the basic construction for $N \subset M$, i.e. it is generated by M and e_1 the projection onto the vacuum subspace for N. For $e_1 x e_1 = E(x) e_1$ defines a faithful conditional expectation E of M onto N (faithful by the inequality). If we compute the commutant of $(M, e_1)''$ and use Haag duality to replace I^c by I, we are reduced to proving that $A \cong M \cap (\mathbf{C}_1^c)' = N$. But E restricts to a homomorphism of A onto N, which by faithfulness forces $A = N$.

As illustrations of this method, we prove that $SU(2)$ at level 2 and 4, $SU(4)$ at level 2 and $\mathrm{Diff}(S^1)$ for $c = 1/2$ all lead to finite index subfactors. We also show that Haag duality fails for $U(4, \mathbf{H})$ at level one: it is simply connected but not simply laced. (In this connection, we predict that the subfactors for the non-vacuum representations of F_4 and G_2 at level one should have index $4\cos^2 \pi/5$.)

(a) The branching rules for $SU(2)_2 \times SU(2)_2 \subset SU(4)_1$ are:

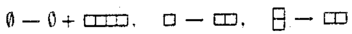


(b) The first four branching rules for $SU(2)_4 \times SU(4)_2 \subset SU(8)_1$ are:

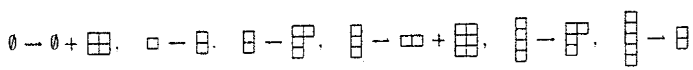


(All others can be obtained by central automorphisms.)

(c) The branching rules for $SU(2)_4 \subset SU(3)_1$ are:



(d) The branching rules for $SU(4)_2 \subset SU(6)_1$ are:



(e) The branching rules for $SU(2)_2 \times \mathrm{Diff}\, S^1_{1/2} \subset SU(2)_1 \times SU(2)_1$ are:

$$\emptyset \otimes \emptyset \to \emptyset \otimes V(1/2, 0) + \square\square \otimes V(1/2, 1/2), \quad \emptyset \otimes \square \to \square \otimes V(1/2, 1/16)$$

where $V(c, h)$ is the representation with central charge c and conformal dimension h.

We shall just use these rules to prove that the subfactors have finite index. (In fact their indices have to be 1, 2, 3 or 4, although we do not prove this.) The vacuum representations for $SU(2)_2$ have diagrams $\emptyset$ and $\square\square$, so from (a) we see that $\square$ has to give a subfactor of index 2. From (c) and (d) we see that the representations of $SU(2)_4$ with diagrams $\emptyset$, $\square\square$ and $\square\square\square\square$ and those of $SU(4)_2$ with diagrams $\emptyset$, $\boxplus$ and $\boxplus$ have finite index. From the branching rules in (b), all representations of $SU(2)_4$ and $SU(4)_2$ have finite index. Next we deal with $\mathrm{Diff}\, S^1_{1/2}$. Taking the self-dual Clifford algebra $\mathrm{Cliff}_M(L^2(S^1))$ generated by a single real Fermi field on the circle in the Fock representation π_P corresponding to Hardy space, this Fock space breaks up as the direct

sum of the projective representations π_0 on $V(1/2, 0)$ and and π_1 on $V(1/2, 1/2)$ of $\mathrm{Diff}\, S^1_{1/2}$ (see [Kac]). Now the even self-dual algebra $\mathrm{Cliff}^+_M(L^2(S^1))$ (i.e. the fixed point algebra under the grading) acts irreducibly on these summands by π_0 and π_1. On the other hand the vacuum vector in $V(1/2, 0)$ is cyclic for both $\mathrm{Diff}_I\, S^1$ and $\mathrm{Cliff}^+(L^2(I))$ (by the Reeh–Schlieder theorem) and the corresponding modular group is geometric. By Takesaki's theorem, $\pi_0(\mathrm{Cliff}^+_M(L^2(I)))'' = \pi_0(\mathrm{Diff}_I\, S^1)''$. We already know Haag duality is satisfied in this sector. We now argue that these local algebras also coincide in the sector π_1 and that Haag duality holds. Any fermi field $\psi(f)$ with f supported in I^c provides a unitary intertwiner $U = \pi_P(\psi(f))$ between π_0 and π_1 for $A(I) = \mathrm{Cliff}^+_M(L^2(I))$. By definition $U\pi_0(A(I^c))U^* = \pi_1(\psi(f)A(I^c)\psi(f)^*) = pi_1(A(I^c))$ and this extends to the weak closures. So Haag duality is also valid for Cliff^+_M in the non-vacuum sector. On the other hand if φ is a diffeomorphism, then $\pi(\varphi)\pi(\psi(\xi))\pi(\varphi)^* = \pi(\psi(\phi.\xi))$ for any $\xi \in L^2(S^1, \mathbf{R})$ and hence $U\pi_0(\phi)U^* = \pi_1(\phi)$ for any $\phi \in \mathrm{Diff}_I\, S^1$. Thus U transports the results for π_0 to π_1. In this way we see that Haag duality folds for $\mathrm{Diff}\, S^1$ in the Neveu–Schwarz sectors ($h = 0, 1/2$). The Ramond sector($h = 1/16$) must have index 2 from (e) and the fact that $SU(2)_2$ has index 2 for $\square$. Finally Haag duality cannot hold at level one for $U(4, \mathbf{H})$ because of the conformal inclusion $U(4, \mathbf{H}) \subset E_6$. The vacuum representation of E_6 splits into two representations for which Haag duality holds, while the other two level one representations remain irreducible [KW]. So the index for these two must be two.

$\Rightarrow N \subset M_0$ has index $(1+1)^2 = 4$ $N \subset M$ has index 2 $\Rightarrow N = M^{\mathrm{inv}}$

19. Vertex operators — In the loop group picture, for a fixed level Tsuchiya and Kanie have constructed vertex operators infinitesimally for $SU(2)$ [TK] (and in published work for $SU(n)$). These are the matter fields or primary blocks of conformal field theory and move between different sectors. They should be unbounded operator valued distributions that intertwine the reporesentations of both the loop group and diffeomorphism group. Taken together they make the whole theory irreducible and are in some sense in duality with the loop group. The vertex operators are known to exist as formal power series but their distributional properties have not been discussed explicitly. Once it is known that they are unbounded operator valued distribution in the sense of Glimm–Jaffe, for example, then when smeared over a test function supported in I^c they will provide intertwiners for $L_I G$ and $\mathrm{Diff}_I\, S^1$. One could then hope by a careful analysis to match up the braiding of Tsuchiya and Kanie with the braiding provided by the local endomorphisms above. This work is still in progress and will be reported on elsewhere.

For the moment we outline a general philosophy for obtaining a workable analytic picture of vertex operators. The idea, inspired by work of Tsuchiya and Nakanishi, is in keeping with our method of devissage from the free field picture using the GKO construction and conformal inclusions. For simplicity we consider only the case of $SU(n)$, although the construction would apply more generally to simply laced groups and other cases of conformal inclusions. There are n level one representations of $LSU(n)$. As we have seen, we can use loops with a central discontinuity to move between different sectors. Since such a loop may be identified (modulo constant central loops) with a loop with values in the adjoint group $PU(n)$ and since constant central loops act as scalars in any positive energy representation, the direct sum of the level one representations give rise to a positive energy irreducible representation of $LPU(n)$. $\approx G/Z(G)$

The idea of extending representations to loops in $\overline{G} = \mathrm{Ad}G$ is also useful from other points of view. For example at higher levels the representations will group together into orbits under the natural action of the centre of G. The direct sum over each orbit will yield an irreducible positive energy representation of $L\overline{G}$ and hence a new subfactor. When $G = SU(2)$, we expect that the subfactors corresponding to $LSO(3)$ should lead to the D_n series of subfactors of index less than 4 (see [GHJ]).

Now given a one-parameter subgroup of T, i.e. a point α of the weight lattice, we may construct vertex operators by Skyrme's standard blip construction as renormalised limits of loops in LT in the same homotopy class as the subgroup but localised at points on the circle. This is just the standard dual resonance model construction of Fubini–Veneziano. The corresponding quantum field $V_\alpha(z)$ has conformal weight $\|\alpha\|^2/2$. The fundamental weights give rise to the primary blocks of [TK] and the Abelian braiding follows from the techniques of dual models (see [GO], [FLM]).

The dual model techniques involve a statistical mechanical interpretation of bosons as an infinite assembly of harmonic oscillators (Coulomb gas). This allows the computation of n-point functions $< V(z_1)V(z_2)\cdots V(z_n)\Omega, \Omega >$. These can be obtained as limits of one loop amplitudes $\mathrm{Tr}(V(z_1)V(z_2)\cdots V(z_n)q^{L_0})$ for which explicit formulae are known involving theta functions. In particular one can compute the Hilbert–Schmidt norms $\|q_1^{L_0}V(z)q_2^{L_0}\|_2$ explicitly. The method of heat kernel regularisation ([Glimm-Jaffe], Chapter 19) shows that $V(z)$ is an unbounded operator valued distribution. Since $V(z)$ has a branch point, one strictly speaking has a distribution on a finite covering of the circle. In fact if f is a test function on this cover, one knows that $q^{L_0}V(f)q^{L_0}$ is a Hilbert–Schmidt operator. The same is true if we replace f by its derivative, so that $q^{L_0}[L_0, V(f)]q^{L_0}$ is a Hilbert–Schmidt operator. On the other hand if $D = \log q\, L_0$, we have formally

$$[q^{L_0}, V(f)] = [\epsilon^D, V(f)] = \int_0^1 \epsilon^{Ds}[D, V(f)]\epsilon^{-Ds}\,ds = \int_0^1 \log q\, q^{sL_0}V(f')q^{-sL_0}\,ds.$$

So we obtain $V(f)q^{2L_0} - q^{L_0}V(f)q^{L_0} = \log q \int q^{sL_0}V(f')q^{(1-s)L_0}\,ds$. The Hilbert–Schmidt norm of the integrand can be estimated explicitly and we deduce that $V(f)q^{2L_0}$ is Hilbert–Schmidt. Similarly $q^{2L_0}V(f)$ is Hilbert–Schmidt. Just being bounded operators already implies that $V(f)$ is a closable densely defined operator. The Hilbert–Schmidt bounds give the additional distributional properties.

Now suppose that $LH \subset LG$ is a conformal inclusion and suppose that the level one representations $\mathcal{H}_i$ of LG branch as $\mathcal{H}_i = \oplus V_{ij} \otimes \mathcal{K}_j$, where the $\mathcal{K}_j$'s are irreducible positive energy representations of LH and V_{ij} is a finite–dimensional multiplicity space. A vertex operator $V(z) : \mathcal{H}_i \otimes V_j \to \mathcal{H}_k$ for LG will give rise to a sum of vertex operators for LH. From a knowledge of the restriction map $R_q(G) \to R_q(H)$ between the fusion algebras at the relevant levels, one can predict that all vertex operators of LH will appear in this way. On the other hand the heat kernel estimates for $V_j(z)$ immediately imply similar estimates for the vertex operators for LH so the analytic properties follow. As an important special case one can use the conformal inclusions $SU(n)_m \times SU(m)_n \subset SU(nm)_1$. The branching rules have been determined as particular cases of rank–level duality (see [Tsu-Nak]) and all multiplicities are either 0 or 1. So $\mathcal{H} = \oplus \mathcal{H}_i^n \otimes \mathcal{H}_i^m$ for a level one representation of $LSU(nm)$, where $\mathcal{H}_i^n$ and $\mathcal{H}_i^m$ are representations of $LSU(n)$ and $LSU(m)$ respectively. We may compress a vertex operator between $\mathcal{H}$ and $\mathcal{K}$ to a map between two spaces $\mathcal{H}_i^n$ and $\mathcal{K}_j^n$ by applying a slice map $\mathrm{id}\otimes\omega$ where $\omega(T) = \langle T\xi, \eta\rangle$ for some lowest energy vectors $\xi \in \mathcal{H}_i^m$ and $\eta \in \mathcal{K}_j^m$. From our infinitesimal knowledge of the vertex operators, we expect that $V(z)$ breaks up as a sum $\oplus V_{ij}^n(z) \otimes V_{ij}^m(z)$. We know that q^{L_0} splits as a straight tensor product on the summands (since we have a conformal inclusion), so the estimate for $\|q^{L_0}V_{ij}^n(z)q^{L_0}\|_2$ follows from the estimate for $q^{L_0}V(z)q^{L_0}$ and the fact that $|\langle V_{ij}^m(z)q^{L_0}.\xi q^{L_0}\eta\rangle|$ is a known monomial in $|z|$ and q. It seems highly probable that other properties of the higher level vertex operators can be established using conformal inclusions. It is worth noting in this context that if $LH \subset LG$ is a conformal inclusion, then the compression of a loop in L_IG will provide an intertwiner for $L_{I^c}H$ between different subrepresentations of a given restriction. This provides another way of obtaining explicit intertwiners.

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