Some notes on CW-orbispaces (work in progress)

André Henriques

December 14, 2005

1 Orbispaces

1.1 Sheaves

Let Top_c denote the category of compact spaces. Declare a collection $\{f_i : V_i \to T\}_{i\in I}$ to be a cover if I is finite and if $\bigcup f_i(V_i) = T$. This defines a Grothendieck topology $\mathcal{T}$ on Top_c (for an introduction to Grothendieck topologies, we refer the reader to the first 10 pages of [5]). A $\mathcal{T}$-sheaf on Top_c is a contravariant functor $F : \mathsf{Top}_c \to \mathsf{Sets}$ such that for every $\mathcal{T}$-cover $\{V_i\}$ of a space $T \in \mathsf{Top}_c$, the map [qsf]

$$F(T) \;\longrightarrow\; \varprojlim \Big[\coprod F(V_{ij}) \leftleftarrows \coprod F(V_i) \Big] \tag{1}$$

is an isomorphism of sets. Here V_{ij} denotes the fibered product $V_i \times_T V_j$.

Example-Definition 1 Let X be an arbitrary topological space. Then the *Yoneda functor*

$$\mathrm{Y}(X) : T \mapsto \mathrm{Hom}(T, X)$$

is a $\mathcal{T}$-sheaf. Indeed, let $\{V_i \to T\}$ be a $\mathcal{T}$-cover. An element in the RHS of (1) is a collection of maps $f_i : V_i \to X$ such that $f_i|_{V_{ij}} = f_j|_{V_{ij}}$. These descend to a map f defined on $\mathrm{colim}(\coprod V_{ij} \rightrightarrows \coprod V_i)$. This colimit is compact and admits a bijective map to T, it is therefore homeomorphic to T. We have produced a map $f : T \to X$, i.e. an element of the LHS of (1).

If X is a compactly generated topological space, namely if it's the colimit of its compact subspaces, then one can recover X from $\mathrm{Y}(X)$. Indeed, if $F = Y(X)$, then the underlying set of X is just $F(pt)$. The topology of X is then the finest one such that for all sheaf map $\mathrm{Y}(T) \to F$, $T \in \mathsf{Top}_c$, the maps $T = \mathrm{Y}(T)(pt) \to F(pt)$ are continuous. One also checks that sheaf maps $\mathrm{Y}(X) \to \mathrm{Y}(X')$ necessarily come from continuous map $X \to X'$. Thus, we have the following version of the Yoneda lemma:

Lemma 2 *The functor* Y *provides a fully faithful embedding of the category of compactly generated topological spaces into the category of $\mathcal{T}$-sheaves on Top_c.* □

Note that the idea of using $\mathcal{T}$-sheaves as a replacement for topological spaces is not new. It is for example almost equivalent to Spanier's quasi-topologies [3].

We shall sometimes extend the notion of $\mathcal{T}$-cover to all compactly generated spaces. In that case, a $\mathcal{T}$-cover of X will be a collection of maps $\{V_i \to X\}_{i\in I}$ such that for every compact subspace $T \subset X$, there exists a finite subset $I' \subset I$ such that $\{V_i \times_X T \to T\}_{i\in I'}$ form a $\mathcal{T}$-cover of T. From now on, all our topological spaces will be assumed to be compactly generated and we shall use the conventions of [4].

Given a CW-complex X, we can replace it by the corresponding sheaf $\mathrm{Y}(X)$. And just like X is the colimit of its skeleta $X^{(n)}$, the sheaf $\mathrm{Y}(X)$ is the colimit of the $\mathrm{Y}(X^{(n)})$. Recall that each skeleton of X is obtained from the previous one by a pushout diagram [cpw]

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & \coprod D^n \\ \downarrow & & \downarrow \\ X^{(n-1)} & \longrightarrow & X^{(n)}. \end{array} \tag{2}$$

A particular feature of the topology $\mathcal{T}$ (not shared by the "open covers" topology) is that (2) induces a pushout of sheaves [cqv]

$$\begin{array}{ccc} \coprod \mathrm{Y}(S^{n-1}) & \longrightarrow & \coprod \mathrm{Y}(D^n) \\ \downarrow & & \downarrow \\ \mathrm{Y}(X^{(n-1)}) & \longrightarrow & \mathrm{Y}(X^{(n)}). \end{array} \tag{3}$$

Lemma 3 *The diagram (3) is a pushout of $\mathcal{T}$-sheaves.*

Proof. Let F denote the pushout of $\mathrm{Y}(X^{(n-1)}) \leftarrow \coprod \mathrm{Y}(S^{n-1}) \rightarrow \coprod \mathrm{Y}(D^n)$, and α the map $F \rightarrow \mathrm{Y}(X^{(n)})$. An element of $F(T)$ is represented by a $\mathcal{T}$-cover $\{V_1, V_2 \rightarrow T\}$ and three compatible elements

$$f_1 \in \mathrm{Y}(X^{(n-1)})(V_1), \quad f_2 \in \coprod \mathrm{Y}(D^n)(V_2), \quad f_{12} \in \coprod \mathrm{Y}(S^{n-1})(V_{12}).$$

In other words, it consists of three compatible maps $f_1 : V_1 \rightarrow X^{(n-1)}$, $f_2 : V_2 \rightarrow \coprod D^n$, and $f_{12} : V_{12} \rightarrow \coprod S^{n-1}$. The map α then sends the triple $(f_1, f_2, f_{12}) \in F(T)$ to the function $f \in \mathrm{Y}(X^{(n)})(T)$ given by

$$f(t) = \begin{cases} f_1(t) & \text{if} \quad x \in V_1 \\ f_2(t) & \text{if} \quad x \in V_2. \end{cases}$$

The inverse $\alpha^{-1} : \mathrm{Y}(X^{(n)})(T) \rightarrow F(T)$ then assigns to f the $\mathcal{T}$-cover [tcV]

$$\left\{V_1 = f^{-1}(X^{(n-1)}), V_2 = f^{-1}(\textstyle\coprod D^n)\right\} \tag{4}$$

and the functions $f_1 = f\big|_{X^{(n-1)}}, f_2 = f\big|_{\coprod D^n}, f_{12} = f\big|_{\coprod S^{n-1}}$. Note that (4) is only a $\mathcal{T}$-cover of T, and typically doesn't refine to an open cover. □

1.2 Stacks

Let Gpds denote the 2-category of groupoids (see [1] for an introduction to 2-categories and bicategories). Given a group G, let $\mathsf{E}G$ denote the groupoid with G as object set, and with exactly one morphism between any two objects. Note that $\mathsf{E}G$ is equivalent to the trivial groupoid, and that it posesses a free action of G. Let us now define $\mathsf{B}G := \mathsf{E}G/G$. This groupoid now has just one object, and G many morphisms.

Definition 4 [fsk] *A $\mathcal{T}$-stack on Top_c is a contravariant functor $F : \mathsf{Top}_c \rightarrow \mathsf{Gpds}$ such that for any $\mathcal{T}$-cover $\{V_i\}$ of a space T, the map* [qsc]

$$F(T) \;\longrightarrow\; \varprojlim\Big[\coprod F(V_{ijk}) \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \coprod F(V_{ij}) \;\substack{\leftarrow\\\leftarrow}\; \coprod F(V_i)\Big] \tag{5}$$

is an equivalence of groupoids. Here $V_{ij} = V_i \times_T V_j$, $V_{ijk} = V_i \times_T V_j \times_T V_k$, and the limit is taken in the bicategorical sense.

We shall define CW-orbispaces as special kinds of $\mathcal{T}$-stacks on Top_c. To view an ordinary CW-complex X as a CW-orbispace, we take the sheaf $\mathrm{Y}(X) : \mathsf{Top}_c \rightarrow \mathsf{Sets}$ and compose it with the natural embedding $\mathsf{Sets} \rightarrow \mathsf{Gpds}$.

1.2.1 Principal bundles

Example-Definition 5 Given a topological group G, we let BG be the $\mathcal{T}$-stackification of the functor $T \mapsto \mathsf{B}(\mathrm{Hom}(T, G))$.

Given $T \in \mathsf{Top}_c$, the groupoid $BG(T)$ is then the colimit over all $\mathcal{T}$-covers of the RHS of (5). An objects in that groupoid is then by definition a collection of objects in $\mathsf{B}(\mathrm{Hom}(V_i, G))$, and a collection of morphisms in $\mathsf{B}(\mathrm{Hom}(V_{ij}, G))$ satisfying a compatibility condition in $\mathsf{B}(\mathrm{Hom}(V_{ijk}, G))$. In other words, it's just a 1-cocycle with values in $\mathrm{Y}(G)$. A morphism between objects (i.e. 1-cocycles) c and c' is a 0-cochain b, defined on a common refinement, such that $b_i c_{ij} b_j^{-1} = c'_{ij}$. Two such 0-cochain are identified if their restrictions to a finer cover are equal.

Let us say that $G \circlearrowright P \to T$ is a *G-principal $\mathcal{T}$-bundle* if there exists a $\mathcal{T}$-cover $\{f_i : V_i \to T\}$ such that $f_i^* P$ and $G \times V_i$ are homeomorphic as G-spaces over V_i. We then have an equivalence of groupoids [jhc]

$$BG(T) \simeq \big\{G\text{-principal } \mathcal{T}\text{-bundles on } T\big\}. \tag{6}$$

Indeed, given a G-principal $\mathcal{T}$-bundle $P \to T$ with chosen trivializations $\varphi_i : f_i^* P \to G$, the 1-cocycle $c_{ij} : V_{ij} \to G$ is the difference between φ_i and φ_j. Inversely, given a 1-cocycle $c_{ij} : V_{ij} \to G$, we can use it to descend the trivial bundles $G \times V_i$ into a bundle over T. This is a $\mathcal{T}$-bundle since it trivializes when pulled back to the V_i. Thus, we could equivalently have taken (6) as our definition of BG.

We note that G-principal $\mathcal{T}$-bundle are not very different from usual G-principal bundles.

Proposition 6 *Let G be a (finite dimensional) Lie group. Then the notions of G-principal $\mathcal{T}$-bundle and G-principal bundle agree. In other words, every G-principal $\mathcal{T}$-bundle is locally trivial in the usual topology.*

Proof. Let $P \to T$ be a G-principal $\mathcal{T}$-bundle. Then P is a locally compact space with proper G-action. It satisfies the hypothesis of Palais' theorem [2] and thus admits slices. □

For the reader's convenience, we sketch the full argument for $G = \mathbb{R}$. A local trivialization of an $\mathbb{R}$-principal $\mathcal{T}$-bundle $P \to T$ is an $\mathbb{R}$-equivariant map $\varphi : P \to \mathbb{R}$ defined in the neighborhood of a given orbit. Such a φ can be written down explicitely: pick a compactly supported function $f : P \to \mathbb{R}_{\geq 0}$ which is not identically zero on that orbit. Then let $\varphi(x) = \int_{t \in \mathbb{R}} t \cdot f(x - t)/I(x)$, where $I(x) = \int_{t \in \mathbb{R}} f(x - t)$. The proof for other Lie groups G uses similar techniques.

However for general G, the two notions are not the same. For example, if $P \to T$ is a non-trivial G-principal bundle, then $\prod_{i=1}^{\infty} P \to \prod_{i=1}^{\infty} T$ is a $(\prod_{i=1}^{\infty} G)$-principal $\mathcal{T}$-bundle, but it's not locally trivial in the usual topology.

If X is not compact but merely compactly generated, we shall say that $G \circlearrowright P \to T$ is a G-principal $\mathcal{T}$-bundle if it is one when restricted to each compact subspace of X. From now on, whenever we say "bundle", we shall always mean "$\mathcal{T}$-bundle" instead.

1.2.2 Quotient stacks

Given a group G acting on a set X, let $X /\!\!/ G$ denote the groupoid $X \times_G \mathsf{E}G$. The set of objects of $X /\!\!/ G$ is X, and an arrow $x \to y$ is given by a group element g such that $gx = y$. Note that the set of isomorphism classes of objects in $X /\!\!/ G$ is just X/G. If the action of G on X is free, then the constant map $\mathsf{E}G \to *$ induces a natural equivalence of groupoids [xgm]

$$X /\!\!/ G \xrightarrow{\sim} X/G. \tag{7}$$

If X is a groupoid equipped with an action of G, we shall also write $X/\!\!/G := X \times_G \mathsf{E}G$. We then have 2-categorical pullback squares [bEB]

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ X/\!\!/G & \longrightarrow & \mathsf{B}G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{B}G \end{array} \tag{8}$$

One can also define $X/\!\!/G$ by mens of a universal property. Namely, for every groupoid Y equipped with the trivial G-action, we have a bijection [qmm]

$$\big\{\text{maps } X/\!\!/G \to Y\big\} \quad \longleftrightarrow \quad \big\{G\text{-equivariant maps } X \to Y\big\}. \tag{9}$$

We should emphasize that in a G-equivariant map $f : X \to Y$, the group G acts not only on X, Y but also on f. For example, the identity $\mathsf{B}G \to \mathsf{B}G$ corresponds to a G-equivariant map $* \to \mathsf{B}G$ where all the action is concentrated on the functor. Note also that puling back that action along the projection $X/\!\!/G \to \mathsf{B}G$, one then recovers the original G-action on X.

Example-Definition 7 Let G be a topological group acting on a topological space X. Then the *quotient stack* $[X/G]$ is the stackification of the functor $T \mapsto \mathrm{Hom}(T, X)/\!\!/\mathrm{Hom}(T, G)$.

The groupoid $[X/G](T)$ is then equivalent to the following geometrically defined one. Its objects are pairs consisting of a G-principal bundle $P \to T$ and a G-equivariant map $P \to X$. The morphisms are then isomorphisms of principal bundles commuting with the equivariant map to X. Indeed, an object in $[X/G](T)$ consists of a cover $\{V_i\}$, and functions $f_i : V_i \to X$, and $g_{ij} : V_{ij} \to G$ satisfying $g_{ij} f_j = f_i$ and $g_{ij} g_{jk} = g_{ik}$. The g_{ij} can then be used to define P while the f_i give the map $P \to X$.

Similarly, if $\mathrm{Y}(G)$ acts on a stack F, then we define $[F/G]$ as the stackification of the functor $F/\!\!/\mathrm{Y}(G)$. This agrees with the above definition when F is of the form $\mathrm{Y}(X)$. The pullbacks (8) then induce pullbacks of stacks [bCG]

$$\begin{array}{ccc} F & \longrightarrow & pt \\ \downarrow & & \downarrow \\ [F/G] & \longrightarrow & BG. \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Y}(G) & \longrightarrow & pt \\ \downarrow & & \downarrow \\ pt & \longrightarrow & BG \end{array} \tag{10}$$

Once again, pulling back the $\mathrm{Y}(G)$ action on $pt \to BG$ recovers the original action on F.

Note that if $G \circlearrowleft X$ is free, then by (7) the stack $[X/G]$ is equivalent to $\mathrm{Y}(X)/\mathrm{Y}(G)$. If the action is proper, then we also have $[X/G] \simeq \mathrm{Y}(X/G)$.

Example 8 [ghg] For $X = H\backslash G$, we have $[X/G] \simeq BH$. Indeed, the first one is the stackification of the functor $T \mapsto \mathrm{Hom}(T, H)\backslash \mathrm{Hom}(T, G) \times_{\mathrm{Hom}(T,G)} \mathsf{E}\,\mathrm{Hom}(T, G) = \mathrm{Hom}(T, H)\backslash \mathsf{E}\,\mathrm{Hom}(T, G)$ while the second one is the stackification of the functor $T \mapsto \mathrm{Hom}(T, H)\backslash \mathsf{E}\,\mathrm{Hom}(T, H)$. The inclusion $\mathsf{E}\,\mathrm{Hom}(T, H) \to \mathsf{E}\,\mathrm{Hom}(T, G)$ induces an equivalence between these two functors. Their stackifications are therefore also equivalent.

A map from $\mathrm{Y}(X)$ to BG is the same thing as an element in $BG(X)$, which is then equivalent to a G-principal bundle on X. This correspondence carries over to stacks. By a G-principal bundle over a stack F, we shall mean a stack Q equipped with an action of $\mathrm{Y}(G)$, and an isomorphism between $[Q/G]$ and F. Note that this agrees with the notion of principal bundle when F is of the form $\mathrm{Y}(X)$. Indeed, if $P \to X$ is a principal bundle in the usual sense, then $\mathrm{Y}(P)$ carries an action

of $\mathrm{Y}(G)$, and $[\mathrm{Y}(P)/G] = [P/G] = \mathrm{Y}(X)$. Inversely, if Q is a stack as above, then we have a pullback
[sfg]

$$\begin{array}{ccc} Q & \longrightarrow & pt \\ \downarrow & & \downarrow \\ \mathrm{Y}(X) & \longrightarrow & BG. \end{array} \tag{11}$$

Pick a $\mathcal{T}$-cover $\{V_i\}$ of X and then refine it so that the composites $\mathrm{Y}(V_i) \to BG$ are equivalent to the trivial map (i.e. classify trivial G-bundles). We then have

$$\mathrm{Y}(V_i) \times_{\mathrm{Y}(X)} Q = \mathrm{Y}(V_i) \times_{BG} pt \simeq \mathrm{Y}(V_i) \times (pt \times_{BG} pt) = \mathrm{Y}(V_i) \times \mathrm{Y}(G).$$

The stack Q is $\mathcal{T}$-locally of the form $\mathrm{Y}(X \times G)$. It's therefore of the form $Q = \mathrm{Y}(P)$, where P is some G-principal bundle over X.

Example 9 Let $H \to G$ be a group homomorphism. Then the corresponding map $BH \to BG$ classifies the G-principal bundle $[G/H] \to BH$. If H is a closed subgroup, this can also be identified with $\mathrm{Y}(G/H) \to BH$.

As in (9), a map from $[F/G]$ to some stack Y is the same the thing as a $\mathrm{Y}(G)$-equivariant map from F to Y. Indeed by the universal property of stackification, a map $[F/G] \to Y$ is the same thing as a map from $F/\!\!/\mathrm{Y}(G)$ to Y. This in turn is equivalent to a $\mathrm{Y}(G)(T)$-equivariant map $F(T) \to Y(T)$ for each $T \in \mathsf{Top}_c$. Phrased differently, it's a $\mathrm{Y}(G)$-equivariant map from F to Y.

Example 10 Let X be an H-space and G a group. Then a map $[X/H] \to BG$ is the same thing an H-equivariant G-principal bundle on X.

1.2.3 The coarse moduli space

By a point of F, we shall mean an object x in the groupoid $F(pt)$, or equivalently a stack morphism $\mathrm{Y}(pt) \to F$. We shall sometimes write abusively $x \in F$.

Given a point x of F, the group $\mathrm{Aut}_{F(pt)}(x)$ is called the *stabilizer* of x and is denoted by $\mathrm{Stab}(x)$. Of course, this only defines it's underlying set of points. A more correct definition of $\mathrm{Stab}(x)$ is to say that it's the sheaf $T \mapsto \mathrm{Aut}_{F(T)}(x|_T)$, where $x|_T$ denotes the image of x in $F(T)$ under the morphism $F(pt) \to F(T)$ induced by $T \to pt$.

Given a stack F, it's *coarse moduli space* $\tau_0 F$ is the sheafification of the functor $T \mapsto \pi_0(F(T))$, where here π_0 denotes the set of isomorphism classes of objects. One should think of $\tau_0 F$ as the underlying space of F. In other words, $\tau_0 F$ is the thing we obtain after killing all the stabilizers groups.

Example 11 [xtb] The coarse moduli space $\tau_0 BG$ is just a point. Indeed, let $*_T \in BG(T)$ denote the trivial bundle. Any G-principal bundle is $\mathcal{T}$-locally trivial by definition. So for any element $x \in \pi_0(BG(T))$ there exists a cover $\{V_i\}$ of T such that $x|_{V_i} = *_{V_i}$. All the elements of $\pi_0(BG(T))$ thus get identified in the sheafification. For all T we have $\tau_0 BG(T) = \{*\}$, in other words $\tau_0 BG = \mathrm{Y}(pt)$.

Example 12 If X is a G-CW-complex, then $\tau_0([X/G]) = \mathrm{Y}(X/G)$. Since the four functors $[\ \ /G]$, τ_0, $\ /G$, and Y preserve the disjoint unions, pushouts, and colimit used to build G-CW-complexes, it's enough to treat the case $X = D^n \times H\backslash G$. And indeed, we have $[(D^n \times H\backslash G)/G] = D^n \times BH$ by Example 8, and $\tau_0(D^n \times BH) = \mathrm{Y}(D^n)$ by example 11.

1.3 CW-orbispaces

From now on, we will abuse notation and write X instead of $\mathrm{Y}(X)$.

Definition 13 [dwo] *A CW-orbispace is a $\mathcal{T}$-stack X of the form $\varinjlim X^{(n)}$, where each $X^{(n)}$ is obtained from the previous one by a pushout* [POs]

$$\begin{array}{ccc} \coprod_j (S^{n-1} \times BG_j) & \longrightarrow & \coprod_j (D^n \times BG_j) \\ {\scriptstyle \coprod \alpha_j}\big\downarrow & & \big\downarrow \\ X^{(n-1)} & \longrightarrow & X^{(n)}. \end{array} \tag{12}$$

Moreover, all attaching maps $\alpha_j : S^{n-1} \times BG_j \to X^{(n-1)}$ are required to induce closed inclusions of stabilizer groups.

References

[1] Jean Bénabou. Introduction to bicategories. *Reports of the Midwest Category Seminar*, pages 1–77, 1967.

[2] Richard S. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math.*, 73(2):295–323, 1961.

[3] E. Spanier. Quasi-topologies. *Duke Math. J.*, 30(1):1–14, 1963.

[4] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:113–152, 1967.

[5] J. L. Verdier. *Topologies et faisceaux,* Exposé II of *Théorie des topos et cohomologie étale des schémas, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA4)*, volume 269 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1972.