THE WIGHTMAN AND HAAG-KASTLER AXIOMS

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In this lecture we discuss the Wightman axioms, which were a set of axioms which formed the first approach to axiomatizing quantum field theory. After this we treat the Haag-Kastler axioms, which are a translation of the axioms into the language of nets of algebras of local observables. Before we can treat the Wightman axioms, we have to give a short introduction into the mathematics of spacetime, its symmetries and the representation of these symmetries on the Hilbert space of physical states.

1. Spacetime and symmetries

1.1. **Spacetime.** The main observation leading to special relativity is that space and time are not distinct concepts, rather we must consider the 4-dimensional spacetime manifold of all possible pointlike events. Still we can make a distinction between past and future of a pointlike event x: we can divide spacetime into three parts: the forward cone V^+ containing all events which can be causally influenced from x, the backward cone V^- containing events from which an influence on x can come and the complement S, the "space-like region". The boundary is formed by all events which can be reached by signals travelling at the speed of light. This causal decomposition expresses a locality principle: no physical effect can propagate faster than the speed of light.

Date: March 13, 2013.

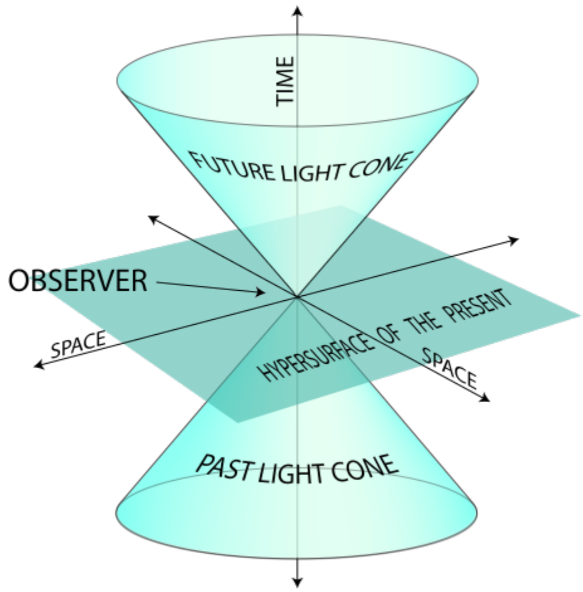

FIGURE 1. Light-cone.

Distances in spacetime are measured by the "Lorentz distance": the squared distance between two points x, x' in spacetime is

$$\Delta x^2 = g_{\mu\nu}(x'^{\mu} - x^{\mu})(x'^{\nu} - x^{\nu}), \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

where we sum over repeated indices. If this is positive we call the points "time-like" separated, if negative "space-like" and is zero "light-like". The space-time manifold equipped with this metric is called Minkowski space and denoted $\mathcal{M}$.

1.2. **Poincaré group.** The invariance group of $\mathcal{M}$ is called the Poincaré group. We can view invariance in 2 different ways. Either we consider maps from $\mathcal{M}$ to $\mathcal{M}$ with respect to a fixed coordinate system, where x is mapped to $x' = gx$ under a diffeomorphism g, this is the active interpretation. Or we consider g as a coordinate transformation, such that x and x' denote the same point in different coordinate systems, this is the passive interpretation. The invariance group consists of all maps (or coordinate transformations) which do not change the Lorentz distance between points. This implies that

$$x'^{\mu} = gx^{\mu} = a^{\mu} + \Lambda^{\mu}_{\nu} x^{\nu} \tag{2}$$

where a^{μ} and Λ^{μ}_{ν} are constant and Λ satisfies

$$g_{\mu\nu}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta} = g_{\alpha\beta}. \tag{3}$$

If we regard Λ^μ_ν and $g_{\mu\nu}$ as 4×4 matrices and Λ^T denotes the transposed matrix we can rewrite this as

$$\Lambda^T \mathbf{g} \Lambda = \mathbf{g} \tag{4}$$

Now a general element $g = (a, \Lambda)$ of the Poincare group, denoted $\mathcal{P}$, consist of a translation 4-vector a^μ and a Lorentz matrix Λ. The (full) Lorentz group can be described as the set of 4×4 matrices satisfying equation 4. This implies

$$(\det \Lambda)^2 = 1 \;\rightarrow\; \det \Lambda = \pm 1 \tag{5}$$
$$(\Lambda^0_0)^2 = 1 + \sum_i (\Lambda^i_0)^2 \;\rightarrow\; \begin{cases} \Lambda^0_0 \geq +1, \text{ or} \\ \Lambda^0_0 \leq -1 \end{cases}$$

The full Lorentz group consists of four disconnected pieces depending on the above sign combinations. The branch with $\det \Lambda = 1$ and $\Lambda^0_0 \geq 1$ will be denoted $\mathcal{L}$ and we will call this the Lorentz group from now on. The other branches can be obtained from $\mathcal{L}$ by reflections in space (or) time.
Fields appearing in classical physics have simple transformation rules under $\mathcal{P}$. Let Φ be a scalar field, V^μ a vector field and W_μ a 1-form, then under $g = (a, \Lambda)$ we have

$$\Phi'(x) = \Phi(g^{-1}x) \tag{6}$$
$$V'^\mu = \Lambda^\mu_\nu V^\nu (g^{-1}x) \tag{7}$$
$$W'^\mu = \tilde{\Lambda}^\nu_\mu W_\nu \tag{8}$$

where $\tilde{\Lambda} = (\Lambda^T)^{-1}$. This can easily be extended to higher tensors.

1.3. **Covering group.** It is important to note that $\mathcal{L}$ is locally isomorphic to $SL(2, \mathbb{C})$. For V^μ a 4-vector, we can form the matrix

$$\hat{V} = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu \sigma_\mu \tag{9}$$

where $\sigma_0 = 1_2$ and σ_i are the Pauli matrices. Note that $\det(\hat{V}) = g_{\mu\nu} V^\mu V^\nu$. Now if α is a complex 2×2 matrix with determinant 1, then the transformation

$$\hat{V} \mapsto \hat{V}' = \alpha \hat{V} \alpha^* \tag{10}$$

gives a matrix with the same determinant. Therefore, if α defines the linear transformation on the components $V'^\mu = \Lambda^\mu_\nu V^\nu$, this Λ must satisfy equation 3.
We have

$$V^\mu = \frac{1}{2} \mathrm{tr}(\hat{V} \sigma_\mu), \; \Lambda(\alpha)^\mu_\nu = \frac{1}{2} \mathrm{tr}(\alpha \sigma_\mu \alpha^* \sigma_\nu) \tag{11}$$

We see that α and $-\alpha$ give the same Λ: the isomorphism is $2:1$. Now $\mathcal{L}$ is not simply connected, but $SL(2, \mathbb{C})$ is. Therefore it is the covering group of $\mathcal{L}$ and we denote it by $\bar{\mathcal{L}}$. Accordingly, we denote the covering group of the Poincaré group by $\bar{\mathcal{P}}$.

1.4. **Poincaré invariance.** Let us consider a physical system whose behaviour we want to study and an observer with instruments capable of making measurements. We can define the the set of possible states of the system and the set of observables which can be measured on it. These states and observables have there physical counterparts in instruments which act as sources of the system (preparing it in a certain state) of detectors of an event (measuring a certain observable).

There are two different types of information in the description of an instrument. Besides the intrinsic construction of the apparatus, we can specify how the equipment is placed with respect to some space-time reference system: we can put it in different positions, rotate it, turn it on at a time of our choice or let it move at a constant velocity. The placement is specified by 10 parameters, they correspond to the 10 paremeters of $\mathcal{P}$. Poincaré invariance of the laws of nature means that the result of an experiment should not depend on the placement: if both source and detector undergo the same shift of placement, the outcome should be the same.

Suppose we have an ideal source which prepares the system in a pure state. For a specified placement in our space-time reference system, this state is described by a ray $\hat{\Psi} = \{\lambda\Psi\}$ in a Hilbert space $\mathcal{H}$ ($\Psi \in \mathcal{H}, \lambda \in \mathbb{C} \setminus \{0\}$). Also, suppose we have an ideal detector which gives a yes-answer in the case our system is in the pure state $\hat{\Phi}$ and no in the orthogonal complement of Φ.

The probability of detecting yes is given by the *ray product*

$$[\hat{\Phi}|\hat{\Psi}] = \frac{|(\Phi, \Psi)|^2}{(\Phi, \Phi)(\Psi, \Psi)} \tag{12}$$

with $(\cdot, \cdot)$ the scalar product in $\mathcal{H}$.

Shifting the placement of the source by a Poincaré transformation $g = (a, \Lambda)$, we denote the state prepared by this source by $\hat{\Psi}_g$. Correspondingly, shifting the detector by $g \in \mathcal{P}$ gives $\hat{\Phi}_g$. Now the invariance means that the probability of "yes" remains the same if both source and detector are shifted by the same group element:

$$[\hat{\Phi}_g|\hat{\Psi}_g] = [\hat{\Phi}|\hat{\Psi}]. \tag{13}$$

If we keep g fixed and let $\hat{\Psi}$ run through all rays, we obtain a map $\hat{T}_g$ corresponding to $g \in \mathcal{P}$

$$\hat{T}_g \hat{\Psi} = \hat{\Psi}_g \tag{14}$$

which leaves the ray product invariant:

$$[\hat{T}_g \hat{\Phi}|\hat{T}_g \hat{\Psi}] = [\hat{\Phi}|\hat{\Psi}] \tag{15}$$

and satisfies

$$\hat{T}_g \hat{T}_{g'} = \hat{T}_{gg'}. \tag{16}$$

A ray transformation may be replaced in many different ways by a transformation of the vectors of $\mathcal{H}$. We have the following theorem, know as the *Wigner unitary-antiunitary theorem*:

Theorem 1. *A ray transformation $\hat{T}$ which preserves the ray product can be replaced by an operator T, which is determined up to an arbitrary phase factor and is either linear and unitary, or antilinear and antiunitary.*

Now in the case of a continous group, antiunitary can be excluded because every group element is the square of another and the square of two antiunitary operators is unitary. So we get for every $g = (a, \Lambda) \in \mathcal{P}$ a unitary operator $U_g = U(a, \Lambda)$ which acts on $\mathcal{H}$ and is determined up to a phase factor. The multiplication law becomes

$$U_g U_{g'} = e^{i\alpha} U_{gg'} \tag{17}$$

where the phasefactor may depend on g, g'. This makes U a *projective representation* of $\mathcal{P}$. Since we still have the freedom to change every operator U_g by a phase factor, we can simplify the phase function $e^{i\alpha}$. We have:

Theorem 2. *Any ray representation of the Poincaré group can, by a suitable choice of phases, be made into an ordinary representation of the covering group $\bar{\mathcal{P}}$.*

Thus the phase function can be removed if we interpret g, g' as elements of $\bar{\mathcal{P}}$ and is reduced to ± 1 if we consider $\mathcal{P}$ itself.

We conclude: Poincaré invariance in quantum theory means that we have a unitary representation of $\bar{\mathcal{P}}$ on $\mathcal{H}$, which describes the effect of Poincaré transformations on the state vectors. The covering group arises because states correspond to rays instead of vectors of $\mathcal{H}$.

1.5. **Irreducible unitary representations of the Poincaré group.** The different irreducible representations of $\bar{\mathcal{P}}$ are classified by Wigner.
Let $U(a)$ denote the representor of a translation by a 4-vector a^μ and $U(\alpha)$ for that of $\alpha \in \bar{\mathcal{L}}$, where α determines a Lorentz transformation $\Lambda(\alpha)$. Now the translation subgroup is commutative and we can write

$$U(a) = e^{iP_\mu a^\mu} \tag{18}$$

where the infinitesimal generator P_μ are commuting self-adjoint operators. The multiplication in $\bar{\mathcal{P}}$ gives us

$$U(\alpha) P^\mu U(\alpha)^{-1} = \Lambda^\mu_\nu P^\nu; \quad P^\mu \equiv g^{\mu\nu} P_\nu \tag{19}$$

Since the P^μ commute, they have a simultaneous spectral decomposition. The spectral values are a subset of a 4-dimensional space (p-space) and we may represent a general vector $\Psi \in \mathcal{H}$ by the set of spectral components Ψ_p:

$$\Psi = \{\Psi_p\} \tag{20}$$

where Ψ_p is a vector in a generalized eigenspace $\mathcal{H}_p$. In the case of a continuous spectrum, we may consider $\mathcal{H}$ as the direct integral of the spaces $\mathcal{H}_p$ with respect to a positive measure

$d\mu$ in p-space.
According to equation 19, the operator $U(\alpha)$ maps $\mathcal{H}_p$ to $\mathcal{H}_{p'}$, where $p' = \Lambda(\alpha)p$. Clearly all spaces $\mathcal{H}_p$ with p-vectors lying in one orbit under $\mathcal{L}$ will be mapped to each other by the action of the $U(\alpha)$, but spaces with p-vectors lying on different orbits will not be mapped to each other by $U(\alpha)$ or $U(a)$. Therefore, in an irreducible representation, the spectrum of the P_μ must be concentrated on a single orbit. This gives use a division of irreducible representation into the classes:

- m_+: Hyperboloid in forward cone; $p^2 = m^2$ and $p^0 \geq 0$.
- 0_+: Surface of forward cone; $p^2 = m^2$ and $p^0 \geq 0$.
- 0_0: The point $p^\mu = 0$.
- κ: Space-like hyperboloid; $p^2 = -\kappa^2$ (κ real).
- m_-: Hyperboloid in backward cone; $p^2 = m^2$ and $p^0 \leq 0$.
- 0_-: Surface of backward cone; $p^2 = 0$ and $p^0 \leq 0$.

Here $p^2 = g_{\mu\nu} p^{mu} p^{nu} = (p^0)^2 - \mathbf{p}^2$. The operator P^0, considered as observable, is interpreted as the total energy of the system. Also, the P^i are the components of the spacial momentum.
We will only consider the first two classes, for the following reason. One of the most important principles of quantum field theory, ensuring the stability of the system, demands that the energy should have a lower bound. This is not the case in the last three classes. We discard the 0_0 class, since this is not of physical interest (all states have zero energy-momentum). Since the p^μ is interpreted as the energy-momentum, the classification parameter m is of course the rest mass.
We would like to point out that the representations can be further classified by spin, but we will not discuss that here.

2. The Wightman axioms

In the 1950's, Wightman and others began to isolate features of quantum field theory which could be stated in mathematical precise terms. This lead to the "Wightman axioms", which we will now discuss.

2.1. Axiom A. Hilbert space and Poincaré group.

1. We consider a Hilbert space $\mathcal{H}$ which carries a unitary representation of $\bar{\mathcal{P}}$.

2. There is a unique state (ray in $\mathcal{H}$), the physical vacuum, which is invariant under all $U(g)$, $g \in \mathcal{P}$.

3. The spectrum of the energy momentum operators P^μ is confined to the forward cone

$$p^2 \geq 0, \quad p^0 \geq 0 \tag{21}$$

2.2. **Axiom B. Fields.**

1. Fields are ”operator valued distributions” over Minkowski space.

   A quantum field $\Phi(x)$ at a point is not a proper observable. It can be regarded as a sesquilinear form on a dense domain $\mathcal{D} \subset \mathcal{H}$: the matrix element$\langle\Psi_1|\Phi(x)|\Psi_2\rangle$ is finite when $\Psi_1, \Psi_2 \in \mathcal{D}$ and depends linearly on Ψ_1, antilinear on Ψ_2.
   To obain an operator on $\mathcal{D}$ we ”smear out” with a smooth function f:

$$\Phi(f) = \int dx\, \Phi(x) f(x) \tag{22}$$

   If f is a test function (i.e. it is in the Schwarz space of test functions), then $\Phi(f)$ is an (unbounded) operator.
2. The domain $\mathcal{D}$ should contain the vacuum and should be invariant under the action of $\bar{\mathcal{P}}$ and the operators $\Phi(f)$

   If we have several fields, each of which has several tensor of spinor components, we must take test functions for each type (index i) and each component (index λ):

$$\Phi(f) = \sum_i \int d^4x\, \Phi^i_\lambda(x) f^{i\lambda}(x) = \sum_i \Phi^i_\lambda(f^{i\lambda}) \tag{23}$$

2.3. **Axiom C. Hermiticity.**

The set of fields contains with each Φ also the Hermitean Φ^*, defined as as sesquilinear form on $\mathcal{D}$ by

$$\langle\Psi_2|\Phi^*(x)|\Psi_1\rangle = \overline{\langle\Psi_1|\Phi(x)|\Psi_2\rangle} \tag{24}$$

2.4. **Axiom D. Transformation properties.**

The fields transform under $\bar{\mathcal{P}}$ as

$$U(a,\alpha)\Phi^i_\lambda(x)U(a,\alpha)^{-1} = M^{(i)\rho}_\lambda(\alpha^{-1})\Phi^{(i)}_\rho(\Lambda(\alpha)x + a) \tag{25}$$

where $M(\alpha)$ is a finite dimensional representation matrix of $\alpha \in \bar{\mathcal{L}}$.

This expression should be written in terms of the smeared out fields to get:

$$\begin{aligned}
U(a,\alpha)\Phi^i_\lambda(f)U(a,\alpha)^{-1} &= M^{(i)\rho}_\lambda(\alpha^{-1}) \int d^4x\, \Phi^{(i)}_\rho(\Lambda(\alpha)x + a) f(x) && (26)\\
&= M^{(i)\rho}_\lambda(\alpha^{-1}) \int d^4x\, \Phi^{(i)}_\rho(x) f(\Lambda(\alpha)^{-1}(x-a) && (27)\\
&= M^{(i)\rho}_\lambda(\alpha^{-1})\, \Phi^{(i)}_\rho(f_{(a,\alpha)}) && (28)
\end{aligned}$$

with $f_{(a,\alpha)} = f(\Lambda(\alpha)^{-1}(x-a))$

2.5. **Axiom E. Causality.** The fields will satisfy either bosonic or fermionic commutation relations: if the supports of test functions f and g are space-like to each other ($x - y$ is a space-like vector for every $x \in \text{supp}f$ and $y \in \text{supp}g$), then either

$$[\Phi^i(f), \Phi(g)] = 0 \tag{29}$$

or

$$[\Phi^i(f), \Phi(g)]_+ = 0 \tag{30}$$

2.6. **Axiom F. Completeness.** We can approximate any operator acting on $\mathcal{H}$ by taking linear combinations of products of the operators $\Phi(f)$.

2.7. **Axiom G. "Time-slice axiom".** There should be a dynamical law which computes the fields at an arbitary time, in terms of the fields in a small time slice

$$\mathcal{O}_{t,\epsilon} = \{x : |x^0 - t| < \epsilon\} \tag{31}$$

Therefore, the completeness axiom should already apply when we restrict the support of the test functions to $\mathcal{O}_{t,\epsilon}$.

Remark 1. The study of the consequences of these axioms is commonly called *axiomatic quantum field theory.*

3. Haag Kastler axioms

3.1. **Net of algebras.** In the above framework, the fields are used to asign to each open region $\mathcal{O}$ of spacetime an algebra $\mathcal{A}(\mathcal{O})$ of operators acting on a Hilbert space $\mathcal{H}$, namely the algebra generated by all $\Phi(f)$ with $\text{supp}f \subset \mathcal{O}$. This suggests that the *net of algebras* $\mathcal{A}$, that is the correspondence

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \tag{32}$$

makes up the intrinsic mathematical description of the theory.

The algebras constructed from the fields are called *polynomial algebras* because their elements are obtained as sums of products $\Phi(f_1)\Phi(f_2)$.... These operators are in general unbounded. We can however go over to bounded operators: given an observable, which is represented by a self-adjoint (possibly unbounded) operator, we can consider it's spectral projections. This can also be done for more general operators, by making a polar decompositon first. In the following, we we take the algebras $\mathcal{A}(\mathcal{O})$ to be an algebra of bounded operators.

We can now ask ourselfs whether the Wightman axioms are equivalent to a theory formulated in terms of a net of algebras of bounded operators. It turns out that there are some difficulties in going from one framework to the other. We will not discuss these, but note that the Wightman axioms are not sufficient to guarantee the existence of a net of local algebras of bounded operators and conversely, that such a net does not guarantee the existence of field system satisfying the Wightman axioms. But for most purposes these difficulties may be ignored, so we ask to what extend the Wightman axioms can be translated into properties of the net of locas algebras?

3.2. **Axiom A. Hilbert space and Poincaré group.** This axiom can go through without changing. We now have algebras $\mathcal{A}(\mathcal{O})$ of operators acting on the hilbert space $\mathcal{H}$, assumptions on $\mathcal{H}$ and the representation of $\mathcal{P}$ remain.

3.3. **Axiom B. Fields.** The definition of the fields suggests an *additivity property*

$$\mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2) \tag{33}$$

where the $\vee$ denotes the operator algebra generated by the algebras $\mathcal{A}(\mathcal{O}_i)$.

3.4. **Axiom C. Hermiticity.** The Hermiticity mean that $\mathcal{A}(\mathcal{O})$ is a $*$-algebra: the algebra $\mathcal{A}(\mathcal{O})$ come with an involution $A \mapsto A^*$ which assigns to each A its Hermitean conjugate.

3.5. **Axiom D. Transformation property.** The tranformation property becomes, in terms of the net of algebras:

$$U(a,\alpha)\mathcal{A}(\mathcal{O})U(a,\alpha)^{-1} = \mathcal{A}(\Lambda(\alpha)\mathcal{O} + a) \tag{34}$$

So the symmetry operations on spacetime map the algebra of operation of one region to the algebra of the transformed region.

3.6. **Axiom E. Causality.** Two observables associated with space-like separated regions are compatible: the measurement of one observable should not influence the measurement of the other. Therefore, the operators representing these observables must commute.

3.7. **Axiom F. Completeness.** This axiom is unchanged: we demand that every operator acting on $\mathcal{H}$ can be appriximated by linear combinations of products of elements of each $\mathcal{A}(\mathcal{O})$.

3.8. **Axiom G. "Time-slice axiom.** Also this axiom is unchanged.

3.9. **Unobservable fields.** It turns out that in quantum field theory there occur observable and unobservable fields. The observable fields generate a net of algebras $\mathcal{A}_{\text{obs}}(\mathcal{O})$ in which the causality principle is

$$[A_1, A_2] = 0 \text{ if } A_i \in \mathcal{A}_{\text{obs}}(\mathcal{O}_i) \text{ and } \mathcal{O}_1 \text{ is space-like to } \mathcal{O}_2 \tag{35}$$

The unobservable fields lead to *superselection rules*. This will be treated later, but the basic idea is that the Hilbert space $\mathcal{H}$ is a direct sum of supspaces $\mathcal{H}_k$ which are called *superselection sectors*. These sectors are distinguished by global properties such as charge.

The observable algebras $\mathcal{A}_{\text{obs}}(\mathcal{O})$ transform each sector into itself, while the unobservable fields connect different sectors. Now for each sector we have a net of operator algebras $\mathcal{A}_{\text{obs}|\mathcal{H}_k}$, which must contain al physically relevant information. The natural explanation for this is that the intrinsic structure of the theory is fully characterized by the algebraic relations in the net of observable algebras. Hence, the basic object to consider is a net of *abstract algebras* instead of their representative operator algebras on $\mathcal{H}$.

We denote the abstract observable algebra of the region $\mathcal{O}$ by $\mathfrak{U}(\mathcal{O})$. We regard these algebras to be defined only for finite regions (open subsets of $\mathcal{M}$ with compact closure). We can then define the algebra of algebra of "all local observables" as

$$\mathfrak{U}_{\mathrm{loc}} = \cup\, \mathfrak{U}(\mathcal{O}) \tag{36}$$

where the union is taken over all finite regions and we define the C^*-algebra of "quasi-local observables" as

$$\mathfrak{U} = \overline{\mathfrak{U}_{\mathrm{loc}}} \tag{37}$$

where the closure is taken in the norm topology. The superselection rules now come up if the net $\mathfrak{U}$ has inequivalent representations of operator algebras acting on a Hilbert space.

If we formulate the theory in terms of abstract algebras we must reconsider the notion of Poincaré invariance. This means now that to $g \in \mathcal{P}$ there correspond an automorphism α_g of the net with the property

$$\alpha_g \mathfrak{U}(\mathcal{O}) = \mathfrak{U}(g\mathcal{O}). \tag{38}$$

A representation of $\mathfrak{U}$ is a homomorphism from $\mathfrak{U}$ tot a net of operator algebras $\pi(\mathfrak{U})$ acting on some Hilbert space. Given a representation π, the automorpism α_g is called *implementable* in π if there exist a unitary operator $U(g)$ acting in the representation space such that

$$U(g)\pi(A)U(g)^{-1} = \pi(\alpha_g A). \tag{39}$$

We can now state axiom A. in terms of the abstract algebra:

the abstract algebra $\mathfrak{U}$ should have an irreducible representation π_0 in which α_g is implementable and which also satisfies Axiom A.2 and A.3.

The other axioms can also be stated in a simple way in the language of the abstract algebras:

a) The theory is characterized by a net of abstract C^*-algebras

$$\mathcal{O} \to \mathfrak{U}(\mathcal{O}) \tag{40}$$

where $\mathcal{O}$ denotes a finite region of $\mathcal{M}$. The self-adjoint elements of $\mathfrak{U}(\mathcal{O})$ are interpreted as observables which can be measured in $\mathcal{O}$.

b) The Poincaré group is realized by a group of automorphism of the net ($\mathcal{P} \ni g \to \alpha_g$) which satisfies

$$\alpha_g \mathfrak{U}(\mathcal{O}) = \mathfrak{U}(g\mathcal{O}). \tag{41}$$

c) The causality is expressed by
   (i) $[\mathfrak{U}(\mathcal{O}_1), \mathfrak{U}(\mathcal{O}_2)] = 0$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated.
   (ii) If $\hat{\mathcal{O}}$ is the causal completion of $\mathcal{O}$ then $\mathfrak{U}(\hat{\mathcal{O}}) = \mathfrak{U}(\mathcal{O})$.

Remark 2. The prototype of a causally complete region is a diamond or double cone.

Remark 3. The idea to base the theory on a net of local algebras corresponding to spacetime region was proposed by Haag. The axioms in terms of the nets of local observables are called the "Haag-Kastler" axioms.