1 Introduction

In this talk we show that given a conformal net $\mathcal{A}$ and a conditional expectation δ and a faithful normal state ϕ we can extend $\mathcal{A}(I)$ to $\mathcal{B}(I)$ via the Jones extenstion. Furthermore, if instead of δ a so-called Q-system is given we can construct δ from it and $\mathcal{B}(I)$ is a chiral extension. This will be proven by the reconstructiontheorem.

1.1 Preliminaries

First some definitions to be able to understand the statement of the reconstructiontheorem.

Definition 1. *A local Mobius covariant net $\mathcal{A}$ is called irreducible if the local von Neumann algebras $\mathcal{A}(I)$ are all factors.*

Definition 2. *Let $\mathcal{A}$ be a local, conformal net on $\mathbb{R}$. A net of inclusions $\pi(\mathcal{A}(I)) \subset \mathcal{B}(I)$ is called a chiral extension of $\mathcal{A}$ if $\mathcal{B}$ induces an irreducible, isotone, conformal, covariant net of von Neumann algebras through $I \in \mathcal{I}$ on a Hilbertspace $H_{\mathcal{B}}$ that is local relative to $\mathcal{A}$, such that the inclusion above is an inclusion of von Neumann algebras where π is a conformal convariant representation of $\mathcal{A}$ on $H_{\mathcal{B}}$.*

Definition 3. *The chiral extension is called irreducible if all local algebras $\mathcal{A}(I)$ and $\mathcal{B}(I)$ are factors and*

$$\pi(\mathcal{A}(I))' \cap \mathcal{B}(I) = \mathbb{C}1.$$

Definition 4. *Let $A \subset B$ be von Neumann algebras with common unit. A completely positive normalized map $\epsilon : B \to A$ satisfying*

$$\epsilon(a^* ba) = a^* \epsilon(b) a, \tag{1}$$

where $a \in A$ and $b \in B$ is called a conditional expectation. A conditional expectation is called normal if it is weakly continuous.

Definition 5. *Let $\mathcal{A} \subset \mathcal{B}$ be a net of inclusions of von Neumann algebras over a partially ordered index set $\mathcal{J}$, then $\mathcal{A} \subset \mathcal{B}$ has a consistent family of conditional expectations*

$$\epsilon^J : \mathcal{B}(J) \to \mathcal{A}(J),$$

if compatible with the inclusions, that is, $\epsilon^{J_1} = \epsilon^{J_2}|_{J_1}$ for $J_1 < J_2$.

Lemma 1. *Pimsner-Popa Let $\epsilon : B \to A$ be a conditional expectation, then there is a $\lambda \in \mathbb{R}_+$ such that:*

$$\epsilon(b_+) \geq \lambda^{-1} b_+ \tag{2}$$

for all positive elements $b_+ \in B$.

Definition 6. *The smallest constant λ in 2 is called the index* $\mathrm{Ind}(\epsilon)$*. If there exists an ϵ of finite index, then there exists a conditional expectation ϵ_0 that minimizes this quantity. This value is called minimal or the Kosaki index of inclusion*

$$[M : N] = \inf_\epsilon \mathrm{Ind}(\epsilon) = \mathrm{Ind}(\epsilon_0).$$

Definition 7. *A localized endomorphism $\bar{\rho}$ is called conjugated to ρ if there exist two isometric intertwiners*

$$r : \mathrm{id} \to \bar{\rho}\rho \quad \textit{and} \quad \bar{r} : \mathrm{id} \to \rho\bar{\rho},$$

such that

$$\bar{r}^*\rho(r) = r^*\bar{\rho}(\bar{r}) = d^{-1}1,$$

for some $d \in \mathbb{R}_+$. The minimum for d that is obtained by varying r and $\bar{r}$ is called the dimension d_ρ of ρ.

With ρ, $\bar{\rho}$, r and $\bar{r}$ as above, $\phi : a \mapsto r^*\bar{\rho}(a)r$ is called the standard left inverse. Now $\epsilon := \rho \circ \phi$ is a conditional expectation and

$$\epsilon(\bar{r}\bar{r}^*) = \rho(\phi(\bar{r}\bar{r}^*)) = \rho(r^*\bar{\rho}(\bar{r}\bar{r}^*)r) = d_\rho^{-2}1 \geq d_\rho^{-2}\bar{r}\bar{r}^*.$$

In fact it can be shown that

$$\mathrm{Ind}\epsilon = d_\rho^2.$$

Lemma 2. *Let ρ be a localized endomorphism of finite dimension, then*

$$[\mathcal{A} : \rho(\mathcal{A})] = d_\rho^2.$$

In the reconstructiontheorem we will make use of the fact that given an inclusion of von Neumann algebras and a conditional expectation we can extend the bigger von Neumann algebra via what is called the Jones extension.

1.2 Jones extension

Given an inclusion of von Neumann algebras $\mathcal{A}_1 \subset \mathcal{A}(I)$ and a conditional expectation $\delta : \mathcal{A}(I) \to \mathcal{A}_1$. If ω is a faithful normal state on $\mathcal{A}(I)$, then $\phi = \omega \circ \delta$ gives a faithful state on $\mathcal{A}_1$. The GNS construction delivers a vector Ψ that is cyclic for $\mathcal{A}(I)$. Let e now be the projection onto the subspace $\mathcal{A}_1\Psi$, then we define:

$$\mathcal{B}(I) = \mathcal{A}(I) \vee \{e\} = (\mathcal{A}(I) \cup \{e\})''.$$

Now $\mathcal{B}(I)$ is called the Jones extension of $\mathcal{A}(I)$ along δ. Now, we are almost done with the preparation for the theorem, except that we will need the notion of the dual canonical endomorphism of a chiral extension.

1.3 Chiral extension

Let us start with a local, conformal net $\mathcal{A}$ and a chiral extension of finite index $\mathcal{B}$, $\pi(\mathcal{A}) \subset \pi^0(\mathcal{B})$ where π^0 is the vacuum representation of $\mathcal{B}$ on $\mathcal{H}_B$. Now π factors over an algebra homomorphism $\iota : \mathcal{A} \to \mathcal{B}$ such that $\pi = \pi^0 \circ \iota$. Fix an interval I. We can find an $\Psi \in \mathcal{H}_B$ which is cyclic and separating for $\iota(\mathcal{A}(I))$ and $\mathcal{B}(I)$. Let J_A and J_B be the modular conjugation with respect to $\iota(\mathcal{A}(I))$ and $\mathcal{B}(I)$, then we define

$$\gamma : \mathcal{B}(I) \to \mathcal{B}(I)$$

$$b \mapsto J_A J_B b J_B J_A,$$

which is called the canonical endomorphism of the chiral extension $\iota(\mathcal{A}) \subset \mathcal{B}$. Now γ factors over an algebrahomomorphism $\bar{\iota} : \mathcal{B}(I) \to \mathcal{A}(I)$. Now the construction of γ and $\bar{\iota}$ involves the choice of an interval I, but we want to have these homomorphisms independent of the choice of I. The following lemma shows that we can extend them for intervals J such that $I \subset J$.

Lemma 3. *Let $\iota(\mathcal{A}) \subset \mathcal{B}$ be a chiral extension. Then there exists for every interval $I \in \mathcal{I}$ a unitary *-homomorphism $\bar{\iota} : \mathcal{B} \to \mathcal{A}$ such that $\iota \circ \bar{\iota}\_\mathcal{B}(J)$ is a canonical endomorphism for $\mathcal{B}(J)$ if $I \subset J$. Furthermore, $\bar{\iota}$ acts trivially on $\mathcal{B}(I)' \cap \iota(\mathcal{A})$.*

Lemma 4. *Let $\iota(\mathcal{A}) \subset \mathcal{B}$ be a chiral extension and γ a possible extension of the canonical endomorphism with respect to an interval I (as in the last lemma). Then the dual canonical endomorphism*

$$\rho = \bar{\iota} \circ \iota$$

is an in I localized endomorphism of the net $\mathcal{A}$.

Definition 8. *A unitary *-homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ is called $\mathcal{AB}$-morphism. The same holds for $\mathcal{BA}$-morphisms, and ι and $\bar{\iota}$ are examples.*

Two sectors $\alpha : \mathcal{A} \to \mathcal{B}$ and $\bar{\alpha} : \mathcal{B} \to \mathcal{A}$ are called conjugate to each other if there exist two isometric intertwiners $v : \mathrm{id}_B \to \alpha \circ \bar{\alpha}$ in $\mathcal{B}$ and $w : \mathrm{id}_A \to \bar{\alpha}\alpha$ in$\mathcal{A}$ such that the equations

$$\alpha(w)^* v = d_\alpha^{-1} 1_B \tag{3}$$

and

$$w^* \bar{\alpha}(v) = d_\alpha^{-1} 1_A \tag{4}$$

are fullfilledl. In this expression is $d_\alpha = [\mathcal{B}(I) : \alpha(\mathcal{A}(I))]^{1/2}$ and we will refer to this as the dimension of α. It can be shown that ι and $\bar{\iota}$ are conjugated to each other.

2 The Reconstructiontheorem

From the previous considerations we know that a chiral extension $\iota(\mathcal{A}) \subset \mathcal{B}$ comes with ρ a dual canonical endomorphism. So if we want to construct a chiral extension from a net of factors $\mathcal{A}$, then necessarily this net $\mathcal{A}$ must have an endomorphism wich can play the role of the dual canonical endomorphism. Also, given this endomorphism we want to build a conditional expectation, so that along this conditional expetation we can extend $\mathcal{A}$ via the Jones extension. If we want this extension be a chiral extension, then necessarily there must be ι and $\bar{\iota}$ conjugated, i.e. there must be isometric intertwiners v and w such that they intertwine the identity with γ and the identity with ρ respectively. To achieve this some extra structure on $\mathcal{A}$ is required, namely, $\mathcal{A}$ should come with a so-called Q-system.

Definition 9. *Let $\mathcal{A}$ be a completely rational, local, conformal net on $\mathbb{R}$, then a triple (ρ, w, x) consisting of a localized endomorphism ρ, the isometric intertwiner*

$$w : \mathrm{id} \to \rho$$

and

$$x : \rho \to \rho^2$$

is called a (dual) Q-system, if w and x satisfy the following conditions:

$$w^* x = d_\rho^{-1/2} 1 = \rho(w^*) x,$$

$$x x^* = \rho(x^*) x$$

and

$$x x = \rho(x) x.$$

Theorem 1. *Let $\mathcal{A}$ be a local, conformal net and ρ a localized endomorphism, such that there exists a unique (up to a phase) isometry $w : \mathrm{id} \to \rho$. Then ρ is the (dual) canonical enormorphism of an irreducible chiral extension $\iota(\mathcal{A}) \subset \mathcal{B}$ if there exists a Q-system (ρ, w, x). This extentension contains a consistent family of conditional expectations ϵ and the index of all local inclusions is constant and equal to the dimension d_ρ. Finally, the extension is local if and only if $\epsilon(\rho, \rho) x = x$.*

Proof. Let ρ be localized in I. Define a conditional expectation δ by

$$\delta : \mathcal{A}(I) \to \mathcal{A}_1 \subset \mathcal{A}(I)$$
$$a \mapsto x^* \rho(a) x.$$

Let $\mathcal{B}(I) = \mathcal{A}(i) \vee \{e\}$ be the Jones extension along this conditional expectation δ, where e is the Jonesprojection. We define

$$\iota : \mathcal{A}(I) \to \mathcal{B}(I),$$

the inclusion homomorphism. Fact: Any element $b \in \mathcal{B}$ can be written as $b = \iota(a) v$, where $a \in \mathcal{A}$ and v an isometry such that $v v^* = e$. From this we construct the conjugated $\mathcal{BA}$-morphism $\bar{\iota}(\iota(a)v) = \rho(a)x$. Also we define $\gamma = \iota \circ \bar{\iota}$. We can check that v intertwines id and γ. The following arguments show then that ι and $\bar{\iota}$ satisfy 3 and 4.

$$\begin{aligned}
&w^*\bar{\iota}(v) = w^*\bar{\iota}(\iota(1)v) = w^*\rho(1)x = d_\rho^{-1/2}1_A\\
&\iota(w^*)v\iota(a)v = \iota(w^*)(\iota\circ\bar{\iota})(\iota(a)v)v = \iota(w^*\rho(a)\bar{\iota}(v))v =\\
&\iota(aw^*\bar{\iota}(v))v = \iota(ad_\rho^{1/2})v = d_\rho^{1/2}\iota(a)v.
\end{aligned}$$

So that indeed ι and $\bar{\iota}$ are conjugated. Now we can construct a conditional expectation for $\mathcal{A}(I) \subset \mathcal{B}(I)$ by:

$$\epsilon : \mathcal{B}(I) \to \mathcal{A}(I),$$

$$b \to w^*\bar{\iota}(b)w.$$

Now let ω be the vacuum state of net $\mathcal{A}$, then $\omega \circ \epsilon$ defines a state on $\mathcal{B}(I)$ that is invariant under ϵ. By the GNS construction arises a faithful representation π^0 with vacuum vector Ω on the Hilbert space $\mathcal{H}_B$. The representation $\pi^0 \circ \iota$ of $\mathcal{A}(I)$ on the Hilbert space $\mathcal{H}_A = \overline{(\pi^0 \circ \iota)(\mathcal{A}(I))\Omega} \subset \mathcal{H}_B$ can be completed to one of $\mathcal{A}$. Thus, ι is also for elements of $\mathcal{A}$ defined.
Now let $\hat{I}$ be another interval. Let $\hat{\rho}$ represent the same sector as ρ but localized in $\hat{I}$. Now define the local algebra

$$\mathcal{B}(\hat{I}) := \iota(\mathcal{A}(\hat{I})u)v,$$

where $u : \rho \to \hat{\rho}$ is a unitary intertwiner. Define $\hat{w} := uw$ then

$$\hat{w}(x) = uw(x) = u\rho(x)w = \hat{\rho}(x)uw = \hat{\rho}(x)\hat{w},$$

hence $\hat{w}$ intertwines id and $\hat{\rho}$ and is contained in $\mathcal{A}(\hat{I})$. Now $\iota(\hat{w}^*u)v$ is in $\mathcal{B}(\hat{I})$. For $a \in \mathcal{A}(\hat{I})$:

$$\begin{aligned}
&\iota(\hat{w}^*u)v(\iota(a)) = \iota(w^*)v(\iota(a)) = \iota(w^*)\rho(\iota(a))v =\\
&\iota(w^*)\iota(\rho)(a)v = \iota(a)\iota(w^*)v = d_\iota^{-1}\iota(a).
\end{aligned}$$

So $\iota(\hat{w}^*u)v$ is a multiple of 1. Hence $\iota(\mathcal{A}(\hat{I})) \subset \mathcal{B}(\hat{I})$. It follows that $\mathcal{B}$ extends $\mathcal{A}$. We need to check that $\mathcal{B}$ is local relative to $\iota(\mathcal{A})$, i.e. $\mathcal{B}(\hat{I})$ commutes with $\mathcal{A}(I)$ when $\hat{I} \cap I = \emptyset$. For this we need: if $a \in \mathcal{A}(I)$ and $\hat{a} \in \mathcal{A}(\hat{I})$ then $\iota(\hat{a}u)v$ must commute with $\iota(a)$:

$$\begin{aligned}
\iota(\hat{a}u)v\iota(a) &= \iota(\hat{a}u)\iota(\rho(a))v\\
&= \iota(\hat{a}u\rho(a))v\\
&= \iota(\hat{a})\iota(\hat{\rho}(a)u)v\\
&= \iota(\hat{a})\iota(a)\iota(u)v \quad \text{due to locality of } a \text{ and } \hat{\rho}\\
&= \iota(a)\iota(\hat{a}))\iota(u)v \quad \text{due to locality of } a \text{ and } \hat{a}\\
&= \iota(a)\iota(\hat{a}u)v.
\end{aligned}$$

Hence $\mathcal{B}$ is local relative to $\iota(\mathcal{A})$. It remains to show that $\mathcal{B}$ has a consistent family of conditional expectations, that the index of all inclusions is constant and equal to the dimension d_ρ. Define

$$\hat{x} = u\rho(u)xu^*$$

and

$$\hat{\bar{\iota}}(\iota(au)v) = \hat{\rho}(a)\hat{x}$$

a canonical endomorphism for $\iota(\mathcal{A}(\hat{I})) \subset \mathcal{B}(\hat{I})$.

EXERCISE: Check that $(\hat{\rho}, \hat{w}, \hat{x})$ is a (dual) Q-system for $\mathcal{A}(\hat{I})$. Check also that the family of conditional expectations

$$\epsilon^{\hat{I}} : \mathcal{B}(\hat{I}) \to \mathcal{A}(\hat{I}), \quad b \mapsto \hat{w}^* \hat{\bar{\iota}}(b)\hat{w}$$

satisfies the consistency requirement.

Now for if $I_1 \subset I_2$ two intervals $\Phi \in \mathcal{H}_B$ and $b \in \mathcal{B}(I_1)$, then

$$b\Phi = (\omega \circ \epsilon_{I1})(b)\Phi = \omega \circ \epsilon_{I2}|_{I1}(b)\Phi$$

which shows the consistency of the associated representation of $\mathcal{B}(I)$ on one and the same Hilbert space $\mathcal{H}_B$, that comes from $\omega \circ \epsilon$.

Now we want to show that the index of all local inclusions is constant and equal to the dimension d_ρ. The Jones projection saturates the Pimsner- Popa inequality. For type II and type III factors ϵ is the minimal conditional expectation, so

$$[\mathcal{B}(I') : \mathcal{A}(I')]^{-1}1 = \epsilon(vv^*)$$

and

$$\epsilon(v) = \hat{w}^* \hat{\bar{\iota}}(v)\hat{w} = \hat{w}^* \hat{\bar{\iota}}(\iota(u^*u)v)\hat{w} = \hat{w}^* \hat{\rho}(u^*)\hat{x}\hat{w},$$

so that

$$\begin{aligned}
\epsilon(vv^*) &= \hat{w}^* \hat{\rho}(u^*)\hat{x}\hat{x}^* \hat{\rho}(u)\hat{w} \\
&= \hat{w}^* \hat{\rho}(u^*)\hat{\rho}(u)uxu^*u^*ux^*u^*\hat{\rho}(u^*)\hat{\rho}(u)\hat{w} \\
&= \hat{w}^* uxx^*u^*\hat{w} \\
&= w^*u^*uxx^*u^*uw \\
&= w^*xx^*w \\
&= d_\rho^{-1}1.
\end{aligned}$$

Finally, if the extension satisfies locality, then $v \in \mathcal{B}(I)$ and $\iota(u)v \in \mathcal{B}(I')$ must commute if $I \cap I' = \emptyset$. Or,

$$\iota(u)vv = v\iota(u)v,$$

or,

$$vv = \iota(u^*)v\iota(u)v.$$

Also $\bar{\iota}(\iota(a)v) = \rho(a)x$, for $a \in \mathcal{A}(I)$, so in particular $\bar{\iota}(v) = x$, so

$$\begin{aligned}
\iota(x)v &= \iota(\bar{\iota}(v))v = \gamma(v)v = vv = \iota(u^*)v\iota(u)v \\
&= \iota(u^*)\iota(\bar{\iota}(\iota(u)v))v = \iota(u^*)\iota(\rho(u)x)v = \iota(u^*)\iota(\rho(u))\iota(x)v \\
&= \iota(u^*\rho(u))\iota(x)v.
\end{aligned}$$

Since $u^*\rho(u)$ is equal to the braiding operator ϵ, we have $\epsilon x = x$. $\square$