CHAPTER 1

Very quick introduction to the conformal group and cft

The world of Conformal field theory is big and, like many theories in physics, it can be studied in many ways which may seem very confusing at first. First, let us make clear what we are going to study.

Chiral CFT	Full CFT
Vertex algebras	Can be built from Chiral CFTs
Conformal nets	Description of interactions
Examples: loop algebras	Modular invariant partition functions

What physicists usually mean when they talk about a CFT is not directly clear. For now, it is good to think of chiral CFTs as the building blocks of a full CFT, with importance to the construction of a Frobenius algebra, giving importance to being well-defined on n-genus Riemann surfaces to describe interactions. Results show that we can construct many examples of Chiral CFTs from current algebras, i.e. certain loop algebras, of which the most simple case we will encounter is the Virasoro algebra. Although full CFTs seem the natural thing to research, it can be de chiral CFTs which give rise to nice and understandable structures. The fact that the partition function is only modular invariant (that is: it is well defined on a n=1 genus Riemann surface) in the full case and not the chiral case fuels the study of partition functions over some sectors of our full CFT, which in some cases give rise to interesting modular forms containing topological information.

1. Conformal transformations

In this quick talk, we will be working mainly in 1+1 dimensional Minkowski space $\mathbb{R}^{1,1}$. However, for the case of conformal transformations, we will set things up in a more general fashion. Pretty much all of the theorems are proven in [1], all credit goes to the author of this book.

DEFINITION 1.1. Given two Lorentzian manifolds (M, g) and (M', g') of dimension n and $U \subset M$, $V \subset M'$ open subsets of M and M', an

element $\phi \in \mathrm{Hom}(U, V)$ of maximal rank is called a *conformal transformation* if $\exists \Omega_\phi \in \mathrm{Hom}(U, \mathbb{R}_+)$ (called a *conformal factor*) s.t.

$$\phi^* g' = \Omega_\phi g$$

Remarks:

- A trivial yet important example we have seen already are isometries of a spacetime (M, g), i.e. maps $\phi : M \circlearrowleft$ preserving the metric. In this case $\Omega_\phi = 1$.
- It will turn out that the stereographic projection is a good conformal transformation between two spaces (a sphere and a real space minus a point).
- The composition of two conformal maps is again conformal, using multiplication of the corresponding composite conformal factors.
- To give a more concrete picture, we can visualize conformal maps as those which locally preserve angles between curves (hence "hoekgetrouwe veldentheorie").

1.1. Conformal Killing fields. First, we shall look at these conformal mappings infinitesimally, by looking at vector fields whose flow generate a conformal transformation. Remember, given a vector field v on a manifold M and a point $a \in M$, it generates a flow ϕ_t for $t \in I_a$. ϕ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\phi(t, a) \right) = v \left(\phi(t, a) \right), \phi(0, a) = a$$

DEFINITION 1.2. The vector field v is called a *conformal Killing field* if there exists an open subset $U \subset \mathbb{R} \times M$ containing $0 \times M$ s.t. its restricted flow $\phi \upharpoonright U$ gives rise to a family of conformal transformations.

THEOREM 1.3. *Let U be a subset as above and let $g_{\mu\nu}$ be the metric g expressed in local coordinates around the point $(0, a)$. Given a conformal Killing vector field, we define the corresponding conformal Killing factor $\kappa : M \to \mathbb{R}$ in analogy as the conformal factor of a global conformal transformation. In terms of local coordinates, κ satisfies*

$$\partial_\nu v_\mu + \partial_\mu v_\nu = \kappa g_{\mu\nu}$$

PROOF. I shall give the idea for the simple case where $g_{\mu\nu}$ is constant, with the Minkowski metric in two dimensions, our main study, in mind. Use $(\phi_t^* g)_{\mu\nu}(a) = g_{ij}(\phi_t(a))\partial_\mu \phi_t{}^i \partial_\nu \phi_t{}^j$. Differentiate at both sides with t. For the more general case, one replaces the partial derivatives with covariant ones, as one can note that in general the change

of a tensor field w.r.t. the flow of a vector field is given by the Lie derivative. $\square$

DEFINITION 1.4. To be formal, a conformal killing factor κ is a function on M such that there exists a conformal Killing vector field ν corresponding to κ, i.e. $\nabla_\mu \nu_\nu + \nabla_\nu \nu_\mu = \kappa g_{\mu\nu}$.

THEOREM 1.5. *(Schottenloher 1.6)* $\kappa : M \to \mathbb{R}$ *is a conformal Killing factor i.f.f.*

$$(n - 2)\partial_\mu \partial_\nu \kappa + g_{\mu\nu}\square\kappa = 0$$

Note that knowledge about κ is directly related to the effect of *local* conformal transformations. Remember, a conformal Killing vector field generates a flow which is a conformal transformation ϕ. The factor κ is the infinitesimal change of our metric under the flow ϕ.

EXERCISE 1.1. Classify the local transformations for $\mathbb{R}^{3,1}$ by looking at κ. You should get a translation, rotation (w.r.t. the metric), dilatation and an extra transformation (called a special conformal transformation). Show that this Lie algebra of transformations can be identified with elements of $\mathrm{Lie}(SO(3 + 1, 1 + 1))$.

Using these conformal killing factors we can classify the *local* conformal transformations. The following result is true for global conformal transformations on 2d Minkowski space (that is, maps which are defined everywhere):

THEOREM 1.6. *(Schottenloher 1.13) A smooth map $\phi = (u, \nu)$: $M \to \mathbb{R}^{1,1}$ on a connected open subset M is conformal i.f.f.*

- $(\partial_x u)^2 > (\partial_x \nu)^2$.
- $(\partial_x u = \partial_y \nu, \partial_y u = \partial_x \nu)$ *or* $\partial_x u = -\partial_y \nu$ *and* $\partial_y u = -\partial_x \nu$.

The orientation-preserving and $\mathbb{R}$-linear conformal transformations Ψ : $M \to M$ can be represented in coordinates by matrices

$$A = A_+(s, t) = \exp(t) \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix},$$

or

$$A = A_-(s, t) = \exp(t) \begin{pmatrix} -\cosh s & \sinh s \\ \sinh s & -\cosh s \end{pmatrix},$$

that is, they can be decomposed in dilatations and boosts.

2. The conformal group

In this chapter we will solely look at the 2-dimensional Minkowski case. This simplifies everything radically as we do not have to do any conformal compactification yet to make global conformal transformations well defined. (note to the reader: tis the work which involves the action of a quasi-global action of a Lie group on a manifold, which is described in the paper of Longo (1993), if you're reading the paper and wondering what is going on). However, we will need some work to classify the conformal transformations on $\mathbb{R}^{1,1}$.

DEFINITION 2.1. The *conformal group* $\mathsf{Conf}(\mathbb{R}^{1,1})$ of 2d Minkowski space is the connected component containing the identity of conformal diffeomorphisms of $\mathbb{R}^{1,1}$ (or in more generality: its conformal compactification).

Remember, we previously found out that the conformal transformations $\phi : M \to \mathbb{R}^{1,1}$ are maps $\phi = (u, v)$ described in the theorem above.

THEOREM 2.2. *(Schottenloher 2.14) For* $f \in C^\infty(\mathbb{R})$, *let* $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$ *be defined by* $f_\pm(x, y) := f(x \pm y)$. *The map*

$$\Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R}^2),$$

$$(f, g) \mapsto \frac{1}{2}(f_+ + g_-, f_+ - g_-)$$

has the following properties:

(1) $\mathsf{im}\,\Phi = (u, v) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) : u_x = v_y, u_y = v_x$.
(2) $\Phi(f, g)$ *is conformal* $\iff f' > 0, g' > 0$ *or* $f' < 0, g' < 0$
(3) $\Phi(f, g)$ *bijective* $\iff f$ *and* g *are bijective*
(4) $\Phi(fh, gk) = \Phi(f, g)\Phi(h, k)$ *for* $f, g, h, k \in C^\infty(\mathbb{R})$

To visualize, we can make a foliation of 2d Minkowski space using the span of null-vectors. The light ray coordinates $\eta_\pm := t \pm x$ are exactly these null coordinates. Let us write $M = \mathcal{L}_+ \times \mathcal{L}_-$, with $\mathcal{L}_\pm = \{\eta | \eta_\pm = 0\}$. This leads to the next exercise.

EXERCISE 2.1. If a map $f : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ preserves these two foliations and their orientations, then it is conformal.

As we can now build up conformal transformations using the description on this foliation, and restring to the connected component of the identity (and orientation preserving), we have the following result:

COROLLARY 2.3. $\mathsf{Conf}(\mathbb{R}^{1,1}) = \mathsf{Diff}_+(\mathbb{R}) \times \mathsf{Diff}_+(\mathbb{R})$.

This definition makes it possible for us to define a chiral CFT.

DEFINITION 2.4. A chiral CFT has a property that, after complexifying, only one of the two parts of $\mathsf{Conf}(\mathbb{R}^{1,1})$ acts non-trivially. In particular, this means that for chiral CFTs we can restrict our QFT to 1 dimension.

Although beautiful, we did not exploit the conformal compactification at all for our Minkowski space and, most notably, we might be missing some transformations. Note that only our global transformations were well-defined on M. The stereographic projection will give us a conformal embedding of $\mathcal{M}$ into $\mathbb{S}^{1,1}$ and this motivates us to continue our study for $\mathsf{Diff}_+(\mathbb{S}) \times \mathsf{Diff}_+(\mathbb{S})$ and, in the chiral case, $\mathbb{S}$ alone. In this definition I have defined the conformal compactification of the real line to be $\mathbb{S}$. However, there is also the notion of the restricted conformal group.

DEFINITION 2.5. In physics usually the term *global conformal transformation* is used to denote an element of the *restricted conformal group*. This is a subgroup of our conformal group $\mathsf{Diff}_+(\mathbb{S}) \times \mathsf{Diff}_+(\mathbb{S})$ which on our compactified light rays is given by the action of $\mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R})$.

This definition will play a role in the definition of conformal nets, as in some papers one firs defines a local Möbis covariant net on spacetime, which is a net with a notion of covariance w.r.t. the Möbius transformations. That is, there exists a unitary representation of a cover $\overline{\mathsf{PSL}}(2,\mathbb{R})$ of $\mathsf{PSL}(2,\mathbb{R})$ on $\mathcal{H}$ s.t., for every interval I we have

$$U(g)\mathcal{A}(I)U(g)^{-1} = (gI), \qquad g \in U \subset \overline{\mathsf{PSL}}(2,\mathbb{R})$$

3. Towards the Virasoro algebra

When we are given a classical symmetry group, such as our conformal group, one is particularly interested in central extensions in order to lift these symmetries from $U(\mathbb{P})$ to $U(\mathbb{H})$. As the conformal group is the connected component of the identity, we can be equally interested in the central extensions of the Lie algebra.

DEFINITION 3.1. The *Witt algebra* $\mathcal{W} := \mathsf{Lie}(\mathsf{Conf}(\mathbb{S}))_{\mathbb{C}} = \mathsf{Vect}(\mathbb{S})_{\mathbb{C}}$, the complexified space of smooth vector fields on $\mathbb{S}$ with the usual commutator bracket[1] for vector fields.

In order to get acquainted with this algebra and recognize familiar parts, we shall choose a basis in which the commutator bracket becomes

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[1]There are in fact two Lie algebra structures on $\mathsf{Vect}(\mathbb{S})$, however they agree up to a sign.

more familiar. As a general vector field $X \in \mathsf{Vect}(\mathbb{S})$ can be written as $X = f\frac{d}{d\theta}$, with $f \in C^\infty(\mathbb{S})$. After choosing a basis $L_n := -i\exp(in\theta)\frac{d}{d\theta} = z^{1-n}\frac{d}{dz}$ for $z = e^{i\theta}$ (noting that our functions are functions on $C^\infty(\mathbb{S}) = \mathsf{Vect}(\mathbb{S})$), we have

$$L_n L_m f = z^{1-n}\frac{d}{dz}(z^{1-m}\frac{d}{dz}f)$$

And with this we can calculate $[L_n, L_m]f$:

(3.1) $$[L_n, L_m]f = L_n L_m f - L_m L_n f$$

(3.2) $$= ((1-m) - (1-n))z^{1-n-m}\frac{d}{dz}f$$

(3.3) $$= (n-m)L_{n+m}f$$

The Virasoro algebra is a proper central extension of the Witt algebra with the Lie algebra $\mathbb{C}$, which arises naturally when we want to lift our projective unitary representation to a unitary representation on our full Hilbert space. Proper in this case means that the exact sequence of Lie algebras does not split. It turns out the Witt algebra we cannot lift to $U(\mathcal{H})$, however this is the case for the Virasoro algebra. The Virasoro algebra has a non-trivial center with generator K. In a representation of this Virasoro algebra, this K should be represented by a scalar multiple of the identity. Hence, representations of the Virasoro algebra are characterized by at least this value c, called the central charge. For completeness, we state the defining commutation relation of the Virasoro algebra:

$$[L_n, L_m] = [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}K$$

$$[L_n, K] = 0 \forall n \in \mathbb{Z}$$

Highest weight representations of the Virasoro algebra are determined by the L_0 eigenvalue of the highest weight state (the conformal dimension, or its minimal energy after identifying L_0 as the Hamiltonian operator) together with the choice of central charge c. A highest weight representation of the Virasoro algebra gives us one easy way for looking at chiral CFTs (remember, we started with $\mathsf{Diff}_+(\mathbb{S})$). Continuing, we might modify our symmetry group and go to general current algebras. This is a subject for a next talk.

Bibliography

[1] Schottenloher, *A mathematical introduction to conformal quantum field theory.* Lecture notes in physics 759, Springer

[2] R. Brunetti and D. Guido and R. Longo, *Modular Structure and Duality in Conformal Quantum Field Theory.* . Commun.Math.Phys, 1993