Borchers' commutation relations theorem

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Abstract

These are notes for the UU student seminar on algebraic quantum field theory of Spring 2013. We give Martin Florig's proof of Borchers' commutation relations theorem.

1 Introduction

Borchers' theorem will be used next week in the proof of the Bisognano-Wichmann theorem.

2 Analyticity of complex functions on the real line

Theorem 2.1 (Symmetry principle)**.** *Let G be an open subset of $\mathbb{C}$ that is symmetric with respect to the real axis and write G_+ for the part that lies in the upper half-plane, G_- for the part that lies in the lower half-plane and G_0 for its intersection with the real axis. Let f_+ and f_- be continuous functions on $G_+ \cup G_0$ and $G_- \cup G_0$ respectively which agree on G_0 and such that f_+ is analytic on G_+ and f_- is on G_-. Then f_+ and f_- glue to an analytic function f on G.*

Proof. We need to prove that f is analytic on G_0. Let D be a connected open subset of G centered around a point on G_0. Then we wish to show that f is analytic on D by showing that for every closed triangle T contained in D we have $\oint_T f(z)\,\mathrm{d}z = 0$. We may assume that T only shares one edge or vertex with G_0, otherwise we could cut T up into smaller triangles which do satisfy this property. $\square$

If z, z_0 lie in G_-, then $\overline{z}$, $\overline{z_0}$ lie in G_+, where f is analytic. So we can develop f as a power series around $\overline{z_0}$, and filling in $\overline{z}$ gives

$$f(\overline{z}) = \sum a_n(\overline{z} - \overline{z_0})^n.$$

Then

$$\overline{f(\overline{z})} = \sum \overline{a_n}(z - z_0)^n,$$

so $\overline{f(\overline{z})}$ is analytic.

$g(z) := f(z) - \overline{f(\overline{z})}$ is analytic on G if f is analytic on G. It is zero on G_0, so it is zero on all of G. So $f(z) = \overline{f(\overline{z})}$ for all $z \in G$.

Theorem 2.2 (Schwarz' reflection principle)**.** *Let f be a continuous function on $G_+ \cup G_0$ that is analytic on G_+ and real on G_0. Then f extends to an analytic function F on G.*

Proof. Define for $z \in G_-$

$$F(z) := \overline{f(\overline{z})}.$$

Then $F(z)$ is analytic on G_- and continuous on $G_- \cup G_0$. Now apply the symmetry principle. □

This is the one-variable case of the edge-of-the-wedge theorem.

3 Borchers' theorem

Recap of Tomita-Takesaki. Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector Ω. Then define two anti-linear operators on the dense subsets $M\Omega$ and $M'\Omega$ of $\mathcal{H}$ respectively by

$$S_0(x\Omega) := x^*\Omega, \qquad S_0(x\Omega) := x^*\Omega$$

for $x \in M$ and $x' \in M'$. These are well-defined because Ω is separating for both M and M'. It turns out that S_0 and F_0 are closeable, and we denote their closures by S and F. Let $\Delta = S^*S$ be the unique positive and self-adjoint operator and J be the unique anti-unitary operator occurring in the polar decomposition of S: $S = J\Delta^{1/2}$. We have $J\Omega = \Omega = \Delta\Omega$, $J^2 = \mathrm{id}$, $J^* = J$, $J\Delta^{1/2} = \Delta^{-1/2}J$ and $\Delta^{it}J = J\Delta^{it}$.

$$F = S^* = (J\Delta^{1/2})^* = \Delta^{1/2}J = J\Delta^{-1/2},$$

so Δ^{-1} is the modular operator of the pair (M', Ω). Now the main theorem of Tomita-Takesaki theory claims that

$$JMJ = M' \qquad \text{and} \qquad \Delta^{it}M\Delta^{-it} = M \text{ for all } t \in \mathbb{R}.$$

Theorem 3.1 (Borchers' theorem)**.** *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector Ω. Let Δ and J be the associated modular operator and modular conjugation respectively. Suppose that $U\colon \mathbb{R} \to B(\mathcal{H})$ is a strongly continuous one-parameter unitary group, with positive generator, that leaves Ω fixed and such that*

$$U(a)MU(-a) \subseteq M \tag{1}$$

for all $a \geq 0$. Then for all $a, t \in \mathbb{R}$,

$$\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi t a})$$

and

$$JU(a)J = U(a)^*.$$

Borchers' original proof used the theory of several complex variables, but Martin Florig gave a much simpler proof using only the theory of one complex variable and this is the one we will give. It is very much in the spirit of the proof of the Reeh-Schlieder theorem that we saw last week. The idea is that we wish to show that the following inner products are equal for all $x \in M$, $x' \in M'$ and $t \in \mathbb{R}$:

$$\langle x'\Omega, \Delta^{it} U(e^{2\pi t} a) \Delta^{-it} x\Omega \rangle = \langle x'\Omega, U(a) x\Omega \rangle.$$

Since the sets $\{x'\Omega : x' \in M'\}$ and $\{x\Omega : x \in M\}$ are dense in $\mathcal{H}$, this will show the first commutation relation. The trick is now to keep x and x' fixed, and see the left hand side as a function $f(t)$ on $\mathbb{R}$. Then the right hand side is nothing but $f(0)$. So we define a function $f(t)$ on $\mathbb{R}$ as the left hand side, we wish to prove that it is constant and then compare its value at 0 with that at an arbitrary $t \in \mathbb{R}$. Again, as we saw last week, the shortest path between two truths in the real domain passes through the complex domain. We will extend f to an analytic function on the entire complex plane and prove that it is bounded. Then by Liouville's theorem, f will be constant, in particular on the real line. Moreover, comparing that value of f at $\frac{i}{2}$ to its value at 0 will give us the second commutation relation also.

Proof. We may assume that $a \geq 0$, since in that case the commutation equations for $-a$ are the adjoints of the equations for a. In the proof of the Reeh-Schlieder theorem we defined our function on the complex plane by defining functions on the upper and the lower half plane, and then gluing these together. What we will do is cut up the complex plane into infinitely many strips of height $\frac{1}{2}$, define functions on them, and then glue these together. Fix $x \in M$ and $x' \in M'$. On the first strip $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \frac{1}{2}\}$ define the following two functions

$$f_U(z) := \langle x'\Omega, \Delta^{iz} U(e^{2\pi z} a) \Delta^{-iz} x\Omega \rangle$$
$$f_V(z) := \langle x'\Omega, \Delta^{iz} V(e^{2\pi z} a) \Delta^{-iz} x\Omega \rangle,$$

where $V(a) := JU(-a)J$. (Recall Stone's theorem: if $U \colon \mathbb{R} \to B(\mathcal{H})$ is a strongly continuous one-parameter unitary group, then there exists a unique self-adjoint operator A such that $U(t) = e^{itA}$ for all $t \in \mathbb{R}$. This can thus be used to extend U to the complex plane.)

Note that V is also a one-parameter group, since $V(a+b) = JU(-(s+t))J = JU(-s)U(-t)J = JU(-s)J^2U(-t)J = V(a)V(b)$. Although J is anti-linear and anti-unitary, because it appears twice in V, $V(a)$ is linear and unitary. Since J and U leave Ω fixed, so does V. If x_0 is the infinitesimal generator of U, then we have

$$\lim_{t \to 0} \frac{U(t)h - h}{t} = ix_0 h.$$

So then we find the infinitesimal generator of V by

$$\begin{aligned}-i\lim_{a\to 0}\frac{JU(-a)Jh-h}{a} &= -i\lim_{a\to 0}\frac{JU(-a)Jh-J^2h}{a}\\ &= -iJ\lim_{a\to 0}\frac{U(-a)Jh-Jh}{a}\\ &= -iJ(-ix_0)Jh.\end{aligned}$$

Now since J is anti-linear, $-iJ(-ix_0)Jh = -i\cdot iJx_0Jh = Jx_0Jh$. So Jx_0J is the infinitesimal generator of V. We have $\langle Jx_0Jh, h\rangle = \langle x_0Jh, Jh\rangle \geq 0$, so Jx_0J is positive. Furthermore, V also satisfies (1). By the main theorem of Tomita-Takesaki theory we namely have $JMJ = M'$, and if $y\in M$ and $y'\in M'$, then for all $a\geq 0$

$$[U(-a)y'U(a), y] = U(-a)[y', U(a)yU(-a)]U(a) = 0$$

by (1). So $U(-a)M'U(a)\subseteq M'$, and again applying $JM'J = M$, gives $V(a)MV(-a)\subseteq M$. So V satisfies all the properties that we assumed of U.

The function f_U is continuous and bounded on S and analytic on its interior. Since V has identical properties to U, the same holds for f_V.

Now define

$$f(z) := \begin{cases} f_U(z-in) & \text{if } n\leq \operatorname{Im} z\leq n+\frac{1}{2} \text{ for some } n\in\mathbb{Z},\\ f_V(z-i(n+\frac{1}{2})) & \text{if } n+\frac{1}{2}\leq \operatorname{Im} z\leq n+1.\end{cases}$$

The strips in which we cut up the complex plane are closed though, so they overlap on the lines $\{z\in\mathbb{C} : \operatorname{Im} z\in \frac{1}{2}+\mathbb{Z}\} + \{z\in\mathbb{C} : \operatorname{Im} z\in\mathbb{Z}\}$. We need to check that $f_U(t+i/2) = f_V(t)$ and $f_V(t+i/2) = f_U(t)$ for all $t\in\mathbb{R}$. We will only prove the first equation, since the second one is done identically.

We have

$$\begin{aligned}\Delta^{i(t+i/2)}U(e^{2\pi(t+i/2)}a)\Delta^{-i(t+i/2)}x\Omega &= \Delta^{it}\Delta^{-1/2}U(-e^{2\pi t}a)\Delta^{-it}\Delta^{1/2}x\Omega\\ &= \Delta^{it}\Delta^{-1/2}U(-e^{2\pi t}a)J\Delta^{-it}x^*\Omega.\end{aligned}$$

For the second equation, we plugged $J^2 = \mathrm{id}$ between Δ^{-it} and $\Delta^{1/2}x\Omega$, used that $J\Delta^{1/2} = S$ and that $S(x\Omega) = x^*\Omega$, and then that Δ^{-it} and J commute. Next, we use that $JV(a) = U(-a)J$ for all $a\in\mathbb{R}$ and that $\Delta^{-1/2}J = S$:

$$\Delta^{it}\Delta^{-1/2}U(-e^{2\pi t}a)J\Delta^{-it}x^*\Omega = \Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Omega.$$

Now, since Δ and V leave Ω fixed, so does the product $\Delta^{it}V(e^{2\pi t}a)^*$. We plug this between Δ^{it} and Ω:

$$\Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Omega = \Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Delta^{it}V(e^{2\pi t}a)^*\Omega.$$

We now see a left-to-right symmetry appearing. Also, by the main theorem of Tomita-Takesaki theory and since V also satisfies (1), we have that $V(e^{2\pi t}a)\Delta^{-it}x^*\Delta^{it}V(e^{2\pi t}a)^*$ is an element of M. Since $S(y\Omega) = y^*\Omega$ for all $y\in M$,

$$\Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Delta^{it}V(e^{2\pi t}a)^*\Omega = \Delta^{it}V(e^{2\pi t}a)\Delta^{-it}x\Delta^{it}V(e^{2\pi t}a)^*\Omega.$$

Now we can remove $\Delta^{it}V(e^{2\pi t}a)^*$ again, and we finally obtain

$$\Delta^{it}V(e^{2\pi t}a)\Delta^{-it}x\Delta^{it}V(e^{2\pi t}a)^*\Omega = \Delta^{it}V(e^{2\pi t}a)\Delta^{-it}x\Omega.$$

This shows that $f_U(t+i/2) = f_V(t)$ for all $t \in \mathbb{R}$. The check $f_V(t+i/2) = f_U(t)$ is done in the exact same way, so we now know that f is well-defined on the whole of $\mathbb{C}$. By the Symmetry principle, f is also analytic on the lines $\{z \in \mathbb{C} : \operatorname{Im} z \in \frac{1}{2} + \mathbb{Z}\} + \{z \in \mathbb{C} : \operatorname{Im} z \in \mathbb{Z}\}$, so f is analytic on the whole of $\mathbb{C}$. Because f_U and f_V are bounded on S, f is bounded on the whole of $\mathbb{C}$, so by Liouville's theorem it must be constant. Also,

$$\langle x'\Omega, U(a)x\Omega\rangle = f_U(0) = f_U(i/2) = f_V(0) = \langle x'\Omega, V(a)x\Omega\rangle = \langle x'\Omega, JU(-a)Jx\Omega\rangle,$$

so the second commutation relation follows. $\square$

4 The Bisognano-Wichmann theorem

Let $(\mathcal{A}, \mathcal{H}, \Omega)$ be a local, Möbius covariant net of von Neumann algebras on the circle. Let $I \in \mathcal{I}$ be an interval on the circle and Δ_I and J_I the modular operator and the modular conjugation associated to $(\mathcal{A}(I), \Omega)$.

Exercise: show, using Borchers' theorem, that $U(\tau_I(s))$ commutes with $z_I(t) := \Delta_I^{it} U(\delta_I(2\pi t))$ for all $s, t \in \mathbb{R}$. Show also (which doesn't use Borchers' theorem) that $U(g) z_I(t) U(g)^* = z_{gI}(t)$ for all g in the Möbius group and $t \in \mathbb{R}$.

Theorem 4.1 (Bisognano-Wichmann theorem)**.** *Let $I \in \mathcal{I}$ be an interval on the circle and Δ_I and J_I the modular operator and the modular conjugation associated to $(\mathcal{A}(I), \Omega)$. Then we have for all $t \in \mathbb{R}$*

$$\Delta_I^{it} = U(\delta_I(-2\pi t))$$

and

$$J_I = U(r_I).$$

References