# Borchers' commutation relations theorem

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## Contents



**Abstract**

These are notes for the UU student seminar on algebraic quantum field theory of Spring 2013. We give Martin Florig's proof of Borchers' commutation relations theorem.

## 1 Introduction

Borchers' theorem will be used next week in the proof of the Bisognano-Wichmann theorem.

## 2 Analytically extending complex functions across the real line

**Theorem 2.1** (Symmetry principle)**.** *Let $G$ be an open subset of $\mathbb{C}$ which intersects the real axis and write $G_+$ for the part that lies in the upper half-plane, $G_-$ for the part that lies in the lower half-plane and $G_0$ for its intersection with the real axis. Let $f_+$ and $f_-$ be continuous functions on $G_+ \cup G_0$ and $G_- \cup G_0$ respectively which agree on $G_0$ and such that $f_+$ is analytic on $G_+$ and $f_-$ is on $G_-$. Then $f_+$ and $f_-$ glue together as an analytic function $f$ on $G$.*

*Proof.* All we need to do is prove that $f$ is analytic on $G_0$. We will do this via Morera's theorem. Let $D$ be a connected open subset of $G$ centered around a point on $G_0$. Then we wish to show that $f$ is analytic on $D$ by showing that for every closed triangle $T$ contained in $D$ we have $\oint_T f(z)\,\mathrm{d}z = 0$. We may assume that $T$ only shares one edge or vertex with $G_0$, otherwise we could cut $T$ up into smaller triangles which do satisfy this property. □

If $z$, $z_0$ lie in $G_-$, then $\overline{z}$, $\overline{z_0}$ lie in $G_+$, where $f$ is analytic. So we can develop $f$ as a power series around $\overline{z_0}$, and filling in $\overline{z}$ gives

$$f(\overline{z}) = \sum a_n(\overline{z} - \overline{z_0})^n.$$

Then

$$\overline{f(\overline{z})} = \sum \overline{a_n}(z - z_0)^n,$$

so $\overline{f(\overline{z})}$ is analytic.

$g(z) := f(z) - \overline{f(\overline{z})}$ is analytic on $G$ if $f$ is analytic on $G$. It is zero on $G_0$, so it is zero on all of $G$. So $f(z) = \overline{f(\overline{z})}$ for all $z \in G$.

**Theorem 2.2** (Schwarz' reflection principle)**.** *Let $f$ be a continuous function on $G_+ \cup G_0$ that is analytic on $G_+$ and real on $G_0$. Then $f$ extends to an analytic function $F$ on $G$.*

*Proof.* Define for $z \in G_-$

$$F(z) := \overline{f(\overline{z})}.$$

Then $F(z)$ is analytic on $G_-$ and continuous on $G_- \cup G_0$. Now apply the symmetry principle. □

This is the one-variable case of the edge-of-the-wedge theorem.

# 3 Borchers' theorem

Let us recall the setup of Tomita-Takesaki theory.

Let $M$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $\Omega$. Then define two anti-linear operators on the dense subsets $M\Omega$ and $M'\Omega$ of $\mathcal{H}$ respectively by

$$S_0(x\Omega) := x^*\Omega, \qquad F_0(x'\Omega) := x'^*\Omega$$

for $x \in M$ and $x' \in M'$. These are well-defined because $\Omega$ is separating for both $M$ and $M'$. It turns out that $S_0$ and $F_0$ are closeable, and we denote their closures by $S$ and $F$. Let $\Delta = S^*S$ be the unique positive and self-adjoint operator and $J$ be the unique anti-unitary operator occurring in the polar decomposition of $S$: $S = J\Delta^{1/2}$. We list a few of their basic properties that will be needed later.

$J\Omega = \Omega = \Delta\Omega$, $J^2 = \mathrm{id}$, $J^* = J$, $J\Delta^{1/2} = \Delta^{-1/2}J$ and $\Delta^{it}J = J\Delta^{it}$. Also,

$$F = S^* = (J\Delta^{1/2})^* = \Delta^{1/2}J = J\Delta^{-1/2},$$

so $\Delta^{-1}$ is the modular operator of the pair $(M', \Omega)$. Now the main theorem of Tomita-Takesaki theory claims that

$$JMJ = M' \qquad \text{and} \qquad \Delta^{it}M\Delta^{-it} = M \text{ for all } t \in \mathbb{R}.$$

**Theorem 3.1** (Borchers' theorem)**.** *Let $M$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $\Omega$. Let $\Delta$ and $J$ be the associated modular operator and modular conjugation respectively. Suppose that $U\colon \mathbb{R} \to B(\mathcal{H})$ is a strongly continuous one-parameter unitary group, that leaves $\Omega$ fixed and such that*

$$U(a)MU(-a) \subseteq M \tag{1}$$

*for all $a \geq 0$. If $P$ is the infinitesimal generator of $U$, and we have $\pm P \geq 0$, then for all $a, t \in \mathbb{R}$,*

$$\Delta^{it} U(a) \Delta^{-it} = U(e^{\mp 2\pi t} a)$$

*and*

$$JU(a)J = U(a)^*.$$

Borchers' original proof used the theory of several complex variables, but Martin Florig gave a much simpler proof using only the theory of one complex variable and this is the one we will give. It is very much in the spirit of the proof of the Reeh-Schlieder theorem that we saw last week. The idea is that we wish to show that the following inner products are equal for all $x \in M$, $x' \in M'$ and $t \in \mathbb{R}$:

$$\langle x'\Omega, \Delta^{it} U(e^{2\pi t} a) \Delta^{-it} x\Omega \rangle = \langle x'\Omega, U(a) x\Omega \rangle.$$

Since the sets $\{x'\Omega : x' \in M'\}$ and $\{x\Omega : x \in M\}$ are dense in $\mathcal{H}$, this will show the first commutation relation. The trick is now to keep $x$ and $x'$ fixed, and see the left hand side as a function $f(t)$ on $\mathbb{R}$. Then the right hand side is nothing but $f(0)$. So we define a function $f(t)$ on $\mathbb{R}$ as the left hand side, we wish to prove that it is constant and then compare its value at 0 with that at an arbitrary $t \in \mathbb{R}$. Again, as we saw last week, the shortest path between two truths in the real domain passes through the complex domain. We will extend $f$ to an analytic function on the entire complex plane and prove that it is bounded. Then by Liouville's theorem, $f$ will be constant, in particular on the real line. Moreover, comparing that value of $f$ at $\frac{i}{2}$ to its value at 0 will give us the second commutation relation also.

*Proof.* We may assume that $P \geq 0$, because if it is negative, then we replace $M$ by $M'$ and $U(a)$ by $U(-a)$. Here, we use that $\Delta^{-1}$ and $J$ are the modular operator and conjugation associated to $M'$.

We may assume that $a \geq 0$, since in that case the commutation equations for $-a$ are the adjoints of the equations for $a$. In the proof of the Reeh-Schlieder theorem we defined our function on the complex plane by defining functions on the upper and the lower half plane, and then gluing these together. What we will do is cut up the complex plane into infinitely many strips of height $\frac{1}{2}$, define functions on them, and then glue these together. Fix $x \in M$ and $x' \in M'$. On the first strip $\mathbb{S}_{1/2} := \{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \frac{1}{2}\}$ define the following two functions

$$f_U(z) := \langle x'\Omega, \Delta^{iz} U(e^{2\pi z} a) \Delta^{-iz} x\Omega \rangle$$
$$f_V(z) := \langle x'\Omega, \Delta^{iz} V(e^{2\pi z} a) \Delta^{-iz} x\Omega \rangle,$$

where $V(a) := JU(-a)J$. (Recall Stone's theorem: if $U \colon \mathbb{R} \to B(\mathcal{H})$ is a strongly continuous one-parameter unitary group, then there exists a unique self-adjoint operator $A$ such that $U(t) = e^{itA}$ for all $t \in \mathbb{R}$. This can thus be used to extend $U$ to the complex plane.)

Note that $V$ is also a one-parameter group, since $V(a+b) = JU(-(s+t))J = JU(-s)U(-t)J = JU(-s)J^2U(-t)J = V(a)V(b)$. Although $J$ is anti-linear and anti-unitary, because it appears an even number of times in $V(a)$, $V(a)$ is linear and unitary. Since $J$ and $U$ leave $\Omega$ fixed, so does $V$. If $x_0$ is the infinitesimal generator of $U$, then we have

$$\lim_{t \to 0} \frac{U(t)h - h}{t} = ix_0 h.$$

So then we find the infinitesimal generator of $V$ by

$$\begin{aligned}
-i \lim_{a\to 0} \frac{JU(-a)Jh - h}{a} &= -i \lim_{a\to 0} \frac{JU(-a)Jh - J^2 h}{a} \\
&= -iJ \lim_{a\to 0} \frac{U(-a)Jh - Jh}{a} \\
&= -iJ(-ix_0)Jh.
\end{aligned}$$

Now since $J$ is anti-linear, $-iJ(-ix_0)Jh = -i \cdot iJx_0Jh = Jx_0Jh$. So $Jx_0J$ is the infinitesimal generator of $V$. We have $\langle Jx_0Jh, h\rangle = \langle x_0Jh, Jh\rangle \geq 0$, so $Jx_0J$ is positive. Furthermore, $V$ also satisfies (1). By the main theorem of Tomita-Takesaki theory we namely have $JMJ = M'$, and if $y \in M$ and $y' \in M'$, then for all $a \geq 0$

$$[U(-a)y'U(a), y] = U(-a)[y', U(a)yU(-a)]U(a) = 0$$

by (1). So $U(-a)M'U(a) \subseteq M'$, and again applying $JM'J = M$, gives $V(a)MV(-a) \subseteq M$. So $V$ satisfies all the properties that we assumed of $U$.

The function $f_U$ is continuous and bounded on $\mathbb{S}_{1/2}$ and analytic on its interior. Since $V$ has identical properties to $U$, the same holds for $f_V$.

Now define the following function with period $i$ on the whole complex plane:

$$f(z) := \begin{cases} f_U(z - in) & \text{if } n \leq \operatorname{Im} z \leq n + \frac{1}{2} \text{ for some } n \in \mathbb{Z}, \\ f_V(z - i(n + \frac{1}{2})) & \text{if } n + \frac{1}{2} \leq \operatorname{Im} z \leq n + 1. \end{cases}$$

The strips in which we cut up the complex plane are closed though, so they overlap on the lines $\{z \in \mathbb{C} : \operatorname{Im} z \in \frac{1}{2} + \mathbb{Z}\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z \in \mathbb{Z}\}$. For $f$ to be well-defined, we need to check that

$$f_U(t + i/2) = f_V(t) \qquad \text{and} \qquad f_V(t + i/2) = f_U(t) \tag{2}$$

for all $t \in \mathbb{R}$. Let us prove the first equality. We have

$$\begin{aligned}
\Delta^{i(t+i/2)} U(e^{2\pi(t+i/2)}a)\Delta^{-i(t+i/2)}x\Omega &= \Delta^{it}\Delta^{-1/2}U(-e^{2\pi t}a)\Delta^{-it}\Delta^{1/2}x\Omega \\
&= \Delta^{it}\Delta^{-1/2}U(-e^{2\pi t}a)J\Delta^{-it}x^*\Omega.
\end{aligned}$$

For the second equation, we plugged $J^2 = \mathrm{id}$ between $\Delta^{-it}$ and $\Delta^{1/2}x\Omega$, used that $J\Delta^{1/2} = S$ and that $S(x\Omega) = x^*\Omega$, and then that $\Delta^{-it}$ and $J$ commute. Next, we use that $JV(a) = U(-a)J$ for all $a \in \mathbb{R}$ and that $\Delta^{-1/2}J = S$:

$$\Delta^{it}\Delta^{-1/2}U(-e^{2\pi t}a)J\Delta^{-it}x^*\Omega = \Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Omega.$$

Now, since $\Delta$ and $V$ leave $\Omega$ fixed, so does the product $\Delta^{it}V(e^{2\pi t}a)^*$. We plug this between $\Delta^{it}$ and $\Omega$:

$$\Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Omega = \Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Delta^{it}V(e^{2\pi t}a)^*\Omega.$$

We now see a left-to-right symmetry appearing. Also, by the main theorem of Tomita-Takesaki theory and since $V$ also satisfies (1), we have that $V(e^{2\pi t}a)\Delta^{-it}x^*\Delta^{it}V(e^{2\pi t}a)^*$ is an element of $M$. Since $S(y\Omega) = y^*\Omega$ for all $y \in M$,

$$\Delta^{it}SV(e^{2\pi t}a)\Delta^{-it}x^*\Delta^{it}V(e^{2\pi t}a)^*\Omega = \Delta^{it}V(e^{2\pi t}a)\Delta^{-it}x\Delta^{it}V(e^{2\pi t}a)^*\Omega.$$

Now we can remove $\Delta^{it}V(e^{2\pi t}a)^*$ again, and we finally obtain

$$\Delta^{it}V(e^{2\pi t}a)\Delta^{-it}x\Delta^{it}V(e^{2\pi t}a)^*\Omega = \Delta^{it}V(e^{2\pi t}a)\Delta^{-it}x\Omega.$$

This shows that $f_U(t+i/2) = f_V(t)$ for all $t \in \mathbb{R}$. The check $f_V(t+i/2) = f_U(t)$ is done in the exact same way, so we now know that $f$ is well-defined on the whole of $\mathbb{C}$. By the Symmetry principle, $f$ is also analytic on the lines $\{z \in \mathbb{C} : \operatorname{Im} z \in \frac{1}{2} + \mathbb{Z}\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z \in \mathbb{Z}\}$, so $f$ is analytic on the whole of $\mathbb{C}$. Because $f_U$ and $f_V$ are bounded on $S$, $f$ is bounded on the whole of $\mathbb{C}$, so by Liouville's theorem it must be constant. The second commutation relation also follows, but now from comparing the value of $f$ at $z = 0$ with that at $z = i/2$:

$$\begin{aligned}
\langle x'\Omega, U(a)x\Omega\rangle &= f_U(0) \\
&= f_U(i/2) \\
&= f_V(0) \\
&= \langle x'\Omega, V(a)x\Omega\rangle = \langle x'\Omega, JU(-a)Jx\Omega\rangle,
\end{aligned}$$

where we used that $f$ is constant for the second equation and the first relation in (2) for the third. □

# 4 The Bisognano-Wichmann theorem

Notation: $S^1_+$ and $S^1_-$ for upper and lower semi-circle respectively.

- Notice that $\delta(t)S^1_+ = S^1_+$ for all $t \in R$. We then also have for all intervals $I \in \mathcal{I}$ that $\delta_I(t)I = I$. Choose namely a $g \in \text{Mob}$ such that $I = gS^1_+$. Then

$$\delta_I(t)I = g\delta(t)g^{-1}I = g\delta(t)S^1_+ = gS^1_+ = I.$$

- We have $R(\pi)\delta(t)R(\pi)^{-1} = \delta(-t)$, so for all intervals $I \in \mathcal{I}$,

$$\delta_{I'}(t) = gR(\pi)\delta(t)R(\pi)^{-1}g^{-1} = g\delta(-t)g^{-1} = \delta_I(-t).$$

- Notice that $\tau(t)S^1_+ \subseteq S^1_+$ for $t \geq 0$ and $\tau(t)S^1_- \subseteq S^1_-$ for $t \leq 0$. So also $\tau_I(t)I \subseteq I$ and $\tau_{I'}(-t)I \subseteq I$ for $t \geq 0$.
- We have $\delta(s)\tau(t) = \tau(e^s t)\delta(s)$ for all $s, t \in \mathbb{R}$. The same equality holds if we give $\delta$ and $\tau$ subscripts $I$.

**Theorem 4.1.** *Let $U$ be a strongly continuous unitary representation of* Mob *such that the infinitesimal generator of the rotation one-parameter group $U(R(\cdot))$ is positive. Then the generators of the one-parameter groups $U(\tau_I(\cdot))$ are positive, and the generators of the one-parameter groups $U(\tau_{I'}(-\cdot))$ are negative.*

Let $(\mathcal{A}, \mathcal{H}, \Omega)$ be a local, Möbius covariant net of von Neumann algebras on the circle. Let $I \in \mathcal{I}$ be an interval on the circle and $\Delta_I$ and $J_I$ the modular operator and the modular conjugation associated to $(\mathcal{H}, \Omega, \mathcal{A}(I))$.

By the above theorem, the one-parameter unitary groups $U(\tau_I(\cdot))$ and $U(\tau_{I'}(-\cdot))$ satisfy the requirements of Borchers' theorem for the algebra $\mathcal{A}(I)$.

Recall that, by locality, for every interval $I \in \mathcal{I}$ we have $\mathcal{A}(I') \subseteq \mathcal{A}(I)'$ and $\mathcal{A}(I) \subseteq \mathcal{A}(I')'$. Haag duality claims that these inclusions are equalities.

**Theorem 4.2** (Haag duality)**.** *For all intervals $I \in \mathcal{I}$ we have $\mathcal{A}(I') = \mathcal{A}(I)'$.*

*Proof.* It is sufficient to prove that $\mathcal{A}(S^1_-) = \mathcal{A}(S^1_+)'$ by Möbius covariance of the net $\mathcal{A}$. If we namely know this, then choose for an arbitrary interval $I \in \mathcal{I}$ an element $g \in \text{Mob}$ such that $I = gS^1_+$. Then

$$\begin{aligned}\mathcal{A}(I') &= \mathcal{A}(gS^1_-)\\ &= U(g)\mathcal{A}(S^1_-)U(g)^*\\ &= U(g)\mathcal{A}(S^1_+)'U(g)^*\\ &= \big(U(g)\mathcal{A}(S^1_+)U(g)^*\big)'\\ &= \mathcal{A}(gS^1_+)' = \mathcal{A}(I)'.\end{aligned}$$

We know by locality that $\mathcal{A}(S^1_-) \subseteq \mathcal{A}(S^1_+)'$. For the reverse inclusion, start by writing $J_+$ for the modular conjugation of $\mathcal{A}(S^1_+)$. Then we know by the main theorem of Tomita-Takesaki theory that conjugating $\mathcal{A}(S^1_+)$ with $J_+$ gives you its commutant. We prove that the same holds if we conjugate $\mathcal{A}(S^1_-)$:

$$\begin{aligned}J_+\mathcal{A}(S^1_-)J_+ &= J_+\mathcal{A}(R(\pi)S^1_+)J_+\\ &= J_+U(R(\pi))\mathcal{A}(S^1_+)J_+U(R(\pi))^*J_+\\ &= U(R(\pi))J_+\mathcal{A}(S^1_+)J_+U(R(\pi))^*\\ &= U(R(\pi))\mathcal{A}(S^1_+)'U(R(\pi))^*\\ &= \big(U(R(\pi))\mathcal{A}(S^1_+)U(R(\pi))^*\big)'\\ &= \mathcal{A}(R(\pi)S^1_+)'\\ &= \mathcal{A}(S^1_-)'.\end{aligned}$$

Here, we used Borchers' theorem for the third equality and the main theorem of Tomita-Takesaki theory for the fourth.

Next, we use the second inclusion $\mathcal{A}(S^1_+) \subseteq \mathcal{A}(S^1_-)'$ given to us by locality, and write

$$\mathcal{A}(S^1_+)' = J_+\mathcal{A}(S^1_+)J_+ \subseteq J_+\mathcal{A}(S^1_-)'J_+ = J_+J_+\mathcal{A}(S^1_-)J_+J_+ = \mathcal{A}(S^1_-). \qquad \square$$

Exercise: show, using Borchers' theorem, that $U(\tau_I(s))$ commutes with $z_I(t) := \Delta_I^{it}U(\delta_I(2\pi t))$ for all $s, t \in \mathbb{R}$. Show also (which doesn't use Borchers' theorem) that $U(g)z_I(t)U(g)^* = z_{gI}(t)$ for all $g$ in the Möbius group and $t \in \mathbb{R}$.

Note that since $\delta_I$ preserves $I$, by Möbius covariance, for all $t \in \mathbb{R}$

$$U(\delta_I(t))\mathcal{A}(I)U(\delta_I(t))^* = \mathcal{A}(\delta_I(t)I) = \mathcal{A}(I).$$

So $x \mapsto U(\delta_I(2\pi s))xU(\delta_I(2\pi s))^*$ is an automorphism of $\mathcal{A}(I)$.

**Theorem 4.3** (Bisognano-Wichmann theorem)**.** *Let $I \in \mathcal{I}$ be an interval on the circle and $\Delta_I$ and $J_I$ the modular operator and the modular conjugation associated to $(\mathcal{A}(I), \Omega)$. Then we have for all $t \in \mathbb{R}$*

$$\Delta_I^{it} = U(\delta_I(-2\pi t)).$$

*Moreover, $U$ extends to an anti-unitary operator of* $\text{Mob}_\pm$ *such that*

$$J_I = U(r_I).$$

*For all $g \in \mathrm{Mob}_\pm$ we again have Möbius covariance*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

*Proof.* Define for all $t$ the unitary operator $z_I(t) := \Delta_I^{it} U(\delta_I(2\pi t))$. We wish to show that this is the identity operator.

To prove that this is a one-parameter unitary group we need to show that for $s,t \in \mathbb{R}$ we have $\Delta_I^{it} U(\delta_I(2\pi s)) = U(\delta_I(2\pi s))\Delta_I^{it}$. Since $x \mapsto U(\delta_I(2\pi s))xU(\delta_I(2\pi s))^*$ is an automorphism of $\mathcal{A}(I)$, $U(\delta_I(2\pi s))$ induces an automorphism of the triple $(\mathcal{H}, \Omega, \mathcal{A}(I))$. By functoriality of the modular operators, we then have $U(\delta_I(2\pi s))\Delta_I^{it} U(\delta_I(2\pi s))^* = \Delta_I^{it}$, as desired.

Next, we show that $U(\tau_I(s))$ commutes with $z_I(t)$ for all $s,t \in \mathbb{R}$. We use that $\delta(s)\tau(t) = \tau(e^s t)\delta(s)$ to write

$$z_I(t)U(\tau_I(s)) = \Delta_I^{it} U(\delta_I(2\pi t))U(\tau_I(s)) = \Delta_I^{it} U(\tau_I(e^{2\pi t}s))U(\delta_I(2\pi t))$$

Now we use that the strongly continuous one-parameter unitary group $U(\tau_I(\cdot))$ satisfies the conditions of Borchers' theorem. Therefore, $\Delta_I^{it} U(\tau_I(e^{2\pi t}s)) = U(\tau_I(s))\Delta_I^{it}$. This shows that $z_I(t)U(\tau_I(s)) = U(\tau_I(s))z_I(t)$. Since also $U(\tau_{I'}(-\cdot))$ satisfies the conditions of Borchers' theorem, also $U(\tau_{I'}(s))$ commutes with $z_I(t)$:

$$\begin{aligned}
\delta(2\pi t)\tau_{S^1_-}(s) &= \delta(2\pi t)R(\pi)\tau(s)R(\pi)^{-1} \\
&= R(\pi)\delta(-2\pi t)\tau(s)R(\pi)^{-1} \\
&= R(\pi)\tau(e^{-2\pi t}s)\delta(-2\pi t)R(\pi)^{-1} \\
&= R(\pi)\tau(e^{-2\pi t}s)R(\pi)R(\pi)^{-1}\delta(-2\pi t)R(\pi)^{-1} \\
&= \tau_{S^1_-}(e^{-2\pi t}s)\delta(2\pi t).
\end{aligned}$$

Because Mob is generated by the elements $\tau_I(s)$ and $\tau_I(s)$, we see that $U(g)$ commutes with Mob for all $g \in$ Mob. We have that again by functoriality of the modular operators $U(g)z_I(t)U(g)^* = z_{gI}(t)$. For this we would need to show that

$$S_{U(g)\mathcal{A}(I)U(g)^*,0} = U(g)S_{\mathcal{A}(I),0}U(g)^*,$$

because then

$$\begin{aligned}
U(g)J_I U(g)^* U(g)\Delta_I^{1/2} U(g)^* &= U(g)J_I\Delta_I^{1/2}U(g)^* \\
&= U(g)S_{\mathcal{A}(I)}U(g)^* \\
&= S_{U(g)\mathcal{A}(I)U(g)^*} \\
&= S_{\mathcal{A}(gI)} \\
&= J_{gI}\Delta_{gI}^{1/2},
\end{aligned}$$

and then we use the uniqueness of polar decomposition. So we see $U(g)$ as inducing a morphism from the triple $(\mathcal{H}, \Omega, \mathcal{A}(I))$ to $(\mathcal{H}, \Omega, \mathcal{A}(gI))$. So from $U(g)z_I(t)U(g)^* = z_{gI}(t)$ we see that $z_I(t) = z_{gI}(t)$ for all $g \in$ Mob. Since Mob acts transitively on the set of all intervals $\mathcal{I}$, we see that $z_I(t)$ is independent of $I$. In particular $z_I(t) = z_{I'}(t)$, and this second operator is equal to $z_I(-t)$ by Haag duality, the fact that $\Delta^{-1}$ is the modular operator of $\mathcal{A}(I)'$ and the fact that $\delta_{I'}(t) = \delta_I(-t)$. So $z_I(2t) = \mathrm{id}$ for all $t \in \mathbb{R}$. □

Let $r\colon S^1 \to S^1$ be the reflection $z \mapsto \overline{z}$ in the real line. For a general interval $I \in \mathcal{I}$, let $g \in \mathrm{Mob}$ be such that $I = gS^1_+$. Then define

$$r_I := grg^{-1}.$$

This is well-defined since $r$ commutes with dilations. Note that $r^2 = \mathrm{id}_{S^1}$. This is an orientation reversing isometry of $S^1$ and is called the reflection associated with $I$. We will add $r$ to Mob as follows. Define a function

$$\mathrm{Ad}(r)\colon \mathrm{Mob} \to \mathrm{Mob}, \qquad g \mapsto rgr^{-1}.$$

Then $\mathrm{Ad}(r) \in \mathrm{Aut}(\mathrm{Mob})$ and $\mathrm{Ad}(r)^2 = \mathrm{id}_{\mathrm{Mob}}$. Now define a group homomorphism $\varphi\colon \{1, r\} \to \mathrm{Aut}(\mathrm{Mob})$ by $r \mapsto \mathrm{Ad}(r)$.

**Definition 4.4.** We define $\mathrm{Mob}_\pm := \mathrm{Mob} \rtimes_\varphi \{1, r\}$.

So as a set, $\mathrm{Mob}_\pm = \mathrm{Mob} \times \{1, r\}$, and the multiplication is

$$\begin{aligned}
(g,1)(g',1) &= (gg',1)\\
(g,1)(g',r) &= (gg',r)\\
(g,r)(g',1) &= (grg'r^{-1},r)\\
(g,r)(g',r) &= (grg'r^{-1},1).
\end{aligned}$$

A representation $U$ of Mob on a Hilbert space $\mathcal{H}$ is called anti-unitary if $U(g)$ is unitary, respectively anti-unitary, if $g$ is orientation preserving, respectively reversing.

**Theorem 4.5.** *Every unitary, positive energy representation $U$ of* Mob *on a Hilbert space $\mathcal{H}$ extends to an anti-unitary representation $\widetilde{U}$ of* $\mathrm{Mob}_\pm$ *on the same Hilbert space $\mathcal{H}$, up to unitary equivalence.*

# References