Virasoro & lattice conformal nets

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We will give two constructions of conformal nets, i.e. local Möbius covariant nets where the action of the Möbius group is enhanced to an action of the full diffeomorphism group of the circle. The first, the Virasoro net, will be the 'smallest' conformal net around: it essentially consists *just* of the action of the diffeomorphisms on the circle. The second example will be the conformal net associated to a lattice, which describes one chiral part of a string moving on a torus.

1 Virasoro conformal net

The Virasoro net with central charge c consists essentially *just* of a projective action of $\mathrm{Diff}^+(S^1)$ with central charge c. Whenever c takes an allowed value, we can construct such a projective action from the representation theory of the Virasoro algebra. From this we then construct a conformal net, the Virasoro net (with central charge c).

1.1 The Virasoro algebra

Since $\mathrm{Diff}^+(S^1)$ is hard to understand, we start by considering the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle, given by $f(e^{i\theta})\frac{\partial}{\partial\theta}$ for real-valued functions $f \in \mathcal{C}^\infty(S^1)$. We complexify this by considering also complex functions, and focus mainly on the dense subalgebra spanned by complex vector fields of the form $z^{n+1}\frac{\partial}{\partial z} = -ie^{in\theta}\cdot\frac{\partial}{\partial\theta} =: L_n$. These vector fields satisfy $[L_n, L_m] = (m-n)L_{n+m}$ and form the so-called Witt algebra. Recall from Laurents talk that this Lie algebra admits a central extension, the Virasoro algebra, which as a vector space is $\mathrm{Span}\{L_n\} \oplus \mathbb{C}\kappa$, with commutator is

$$[L_n, L_m] = (m-n)L_{n+m} + c(L_n, L_m) \qquad\qquad [L_n, \kappa] = [\kappa, \kappa] = 0$$

where c is the 2-cocycle

$$c(L_n, L_m) = \frac{\kappa}{12}(n^3 - n)\delta_{n+m,0}$$

One can check that this is (up to scaling) the only central extension of the Witt algebra. We can extend this cocycle to the whole of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ by

$$\omega\Big(f(e^{i\theta})\frac{\partial}{\partial\theta}, g(e^{i\theta})\frac{\partial}{\partial\theta}\Big) \propto \int_0^{2\pi} D^3\big(f(e^{i\theta})\big)\cdot g(e^{i\theta}) - D^3\big(g(e^{i\theta})\big)\cdot f(e^{i\theta})d\theta \tag{1}$$

where $D = e^{-i\theta}\frac{\partial}{\partial\theta}$. We implicitly embed the (complexified) vector fields in the Virasoro algebra (but not as a Lie algebra). Note that the above cocycle vanishes on vector fields having disjoint supports, so these vector fields still commute as elements of the Virasoro algebra.

We have the following subalgebras of the Virasoro algebra:

- $\mathfrak{t} = \mathbb{C}L_0 \oplus \mathbb{C}\kappa$ a maximal torus.
- $\mathrm{Vir}_+ = \mathrm{Span}\{L_n : n > 0\}$
- $\mathrm{Vir}_- = \mathrm{Span}\{L_n : n < 0\}$
- $\mathfrak{sl}_2 = \mathrm{Span}\{L_{-1}, L_0, L_1\}$. Note that on these generators, the cocycle defining the Virasoro algebra vanishes. In terms of vector fields, the real part (i.e. those vector fields in the span which are of the form $f(e^{i\theta})\frac{\partial}{\partial\theta}$ for f real-valued) gives precisely the vector fields giving rise to the Möbius transformations of the circle.

  For example, $iL_0 = \frac{\partial}{\partial\theta}$ is a true vector field on S^1. The corresponding group element $\exp(i\alpha L_0)$ gives a rotation over angle α. Following the physics terminology, we will say the element L_0 generates rotations (instead of iL_0).

It is important to note that the Lie algebra of the Möbius group sits in the Virasoro algebra *as a Lie algebra.*

1.2 Representations of Virasoro algebra

We will be interested in irreducible representations of the Virasoro algebra which are irreducible, unitary and have positive energy. Unitarity means that there is an inner product such that

$$\langle L_n v, w\rangle = \langle v, L_{-n} w\rangle$$

and positive energy means that the vector field L_0 generating the rotations of the circle acts by a positive operator. Irreducibility and positive energy will motivate us to consider lowest weight modules, i.e. modules generated by a vector v_0 such that $L_n v_0 = 0$ for all $n < 0$.

Remark 1. *The fact that $L_n^* = L_{-n}$ will imply that real vector fields on S^1 will act by skew-adjoint operators: for example L_0 will be self-adjoint, so the vector field $\frac{\partial}{\partial\theta} = iL_0$ will be skew-adjoint, and therefore can be exponentiated to a unitary operator.*

It turns out that for each pair of numbers $c, h \in \mathbb{C}$ (central charge and lowest energy), there is an irreducible lowest weight module $L(c, h)$ of Vir. The condition that the module admits an inner product as above and has positive energy will restrict the possible values of c and h.

1.2.1 Irreps of the Virasoro algebra

We quickly recall the construction of the lowest weight irreducible representations of Vir.

Definition 1. *The Verma module $V(c, h)$ of lowest weight h and central charge c is the universal module containing a lowest weight vector v_0 generating the module such that $L_0 v_0 = h\cdot v_0$, in which κ acts centrally by c. Universality means that any other module with these properties has a unique map from $V(c, h)$.*

Explicitly, using the subalgebras of Vir given above, it is constructed as

$$V(c, h) = U(Vir) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}^-)} \mathbb{C}_{(c,h)}$$

where $\mathbb{C}_{(c,h)}$ is the module where κ acts by c, L_0 acts by h and $L_{n<0}$ acts by 0.

Even more explicitly, $V(c, h)$ is spanned by vectors

$$L_{n_1} L_{n_2} ... L_{n_i} v_0 \qquad\qquad n_1 \geq n_2 \geq ... \geq n_i > 0$$

where v_0 is the lowest weight vector, annihilated by L_n for $n < 0$, $L_0 v_0 = h \cdot v_0$ and $\kappa \cdot v_0 = c \cdot v_0$. Observe that

$$L_0\big(L_{n_1} L_{n_2} ... L_{n_i} v_0\big) = (h + n_1 + ... + n_i) L_{n_1} L_{n_2} ... L_{n_i} v_0$$

so L_0 acts diagonally and has spectrum $h+\mathbb{Z}_{\geq 0}$. In particular, if we want positive energy representations, then $h \geq 0$[1].

Note that L_n increases the eigenvalue of L_0 by n, whereas L_{-n} decreases it by n. Let V_{h+n} be the $h+n$-eigenspace of L_0. It is finite dimensional, with $\dim\{L_0 v = (h+n)v\} = P(n)$ the partition number of n (i.e. the number of ways to write n as a sum of positives, where the order does not matter).

Lemma 1.1. *Let W be a submodule of $V(c,h)$. Then W decomposes as a sum of L_0-eigenspaces: $W = \oplus_{n\geq 0} W \cap V_{h+n}$.*

Definition 2. *Let J be the unique maximal proper submodule of $V(c,h)$. It exists because no proper submodule can contain the lowest weight vector. Taking the sum of all proper submodule then gives a unique maximal proper submodule.*

Let $L(c,h) = V(c,h)/J$ be the unique irreducible module $L(c,h)$ with central charge c and lowest weight h: it is irreducible since any proper submodule would give a proper submodule of $V(c,h)$ not in J and it is unique since any irreducible lowest weight module is a quotient of a Verma module.

The maximal submodule $J(c,h)$ decomposes as $J(c,h) = \bigoplus_n J(c,h) \cap V_{h+n}$. This means that the quotient

$$L(c,h) = \bigoplus_n V_{h+n}/\big(J(c,h) \cap V_{h+n}\big)$$

still carries a diagonal action of L_0, with eigenvalues again contained in $h+\mathbb{Z}_{\geq 0}$. It is not a priori clear if all eigenvalues are still attained, but this turns out to be true (except in the case that $h=0$, in which case we are missing the eigenvalue 1, see below).

1.2.2 Inner product

Lemma 1.2. *When $c,h \in \mathbb{R}$, there is a unique Hermitean form $\langle -,- \rangle$ on $V(c,h)$ such that $L_n^* = L_{-n}$ and $\langle v_0, v_0 \rangle = 1$.*

Proof. It is constructed as follows: first, we have

$$\langle v_0, L_{n_1}...L_{n_i} v_0 \rangle = \langle L_{-n_1} v_0, L_{n_2}...L_{n_i} v_0 \rangle = 0$$

To write down an arbitrary inner product, we use

$$\langle L_{m_1}...L_{m_j} v_0, L_{n_1}...L_{n_i} v_0 \rangle = \langle v_0, L_{-m_j}...L_{-m_1} L_{n_1}...L_{n_i} v_0 \rangle$$

We then write out the right hand side: it will be in the level $h + \sum n - \sum m$-eigenspace of L_0. In particular, it follows that the different eigenspaces of L_0 are orthogonal. If $\sum m = \sum_n$, we end up with some number one can compute.

Note that we need $h \in \mathbb{R}$ to make sure $L_0 = L_0^*$. The condition $c \in \mathbb{R}$ comes for instance from the fact that $\langle L_n v_0, L_n v_0 \rangle \in \mathbb{R}$. Finally, one can show that the above really does define a Hermitean form on $V(c,h)$ (which is automatically unique by the above computation). $\square$

Lemma 1.3. *The maximal proper submodule $J(c,h)$ of $V(c,h)$ is precisely the space*

$$\{v \in V(c,h) : \langle v, - \rangle = 0\}.$$

Proof. It is manifestly a proper submodule. Now suppose there is $X \cdot v_0 \in J(c,h)$ not lying in the above space, so that there is a $Y \cdot v_0$ with

$$1 = \langle X \cdot v_0, Y \cdot v_0 \rangle = \langle Y^* X \cdot v_0, v_0 \rangle$$

This means that $Y^* X \cdot v_0$ has a component along v_0, but (since $Y^* \in Vir$) also $Y^* X \cdot v_0 \in J(c,h)$, since the latter is a submodule. But $J = \bigoplus_{n\geq 0} J \cap V_n$ and we find a vector with component along V_0, so we see that $J \cap V_0 \neq 0$. We conclude that $J = V(c,h)$, which is a contradiction. $\square$

[1] well, actually we want the rotations to have *some* positive generator. Since these act periodically, we can always add or subtract a multiple of 2π, so we really want the spectrum to be bounded from below

Remark 2. *If $h = 0$, we see that $\langle L_{-1}v_0, L_{-1}v_0 \rangle = 0$, so $L_{-1}v_0$ is self-perpendicular. Since it is also perpendicular to all the other eigenspaces, it must lie in $J(c,h)$, so the whole level 1 eigenspace is divided out.*

Since the Hermitean form vanishes on the submodule J, it descends to a Hermitean form on $L(c,h)$ (when $c, h \in \mathbb{R}$). This is the unique Hermitean form on $L(c,h)$ such that $L_n^* = L_{-n}$ and $\langle v_0, v_0 \rangle = 1$.

1.2.3 Unitary and positive energy

We have found a simple module on which L_0 acts positively, together with a Hermitean form. In the end we want the module $L(c,h)$ to have not just a Hermitean form, but we want it to be positive definite.

Lemma 1.4. *If the above (unique) Hermitean form on $L(c,h)$ is positive definite, then $c, h \geq 0$.*

Proof. We have for any n:

$$0 < \langle L_n v_0, L_n v_0 \rangle = \langle v_0, L_{-n} L_n v_0 \rangle = \langle v_0, [L_{-n}, L_n] v_0 \rangle = 2nh + \frac{c}{12}(n^3 - n)$$

Taking $n = 1$, we get that $h \geq 0$ and if $c < 0$, then for large n the right hand side will turn negative. □

A more precise analysis of which c, h can possibly give a positive-definite Hermitean form is done in [1]. It turns out we have the following cases:

- $c \geq 1$ and $h \geq 0$.
- there is $m \in \mathbb{Z}_{\geq 2}$, together with $p = 1, ..., m-1$, $q = 1, ..., p$ such that

$$c = c(m) = 1 - \frac{6}{m(m+1)}$$

and

$$h = h_{p,q}(m) = \frac{\big((m+1)p - mq\big)^2 - 1}{4m(m+1)}$$

On the other hand, all the above cases really do allow for such an inner product [2]. Throughout we will assume we are in one of the above situations.

Now that we have an inner product, we can take the Hilbert space completion $H(c,h)$ of $L(c,h)$. We can then view all operators L_n as unbounded operators on the dense domain $L(c,h)$. $L(c,h)$ consists of the so-called finite energy representations, which means that it consists only of vectors which decompose into finitely many L_0-eigenstates.

1.3 Action of $\mathrm{Diff}^+(S^1)$

We have seen how to construct irreducible positive energy representations of the Virasoro algebra and which ones of them allow for an inner product such $L_n^* = L_{-n}$. We now construct a net with central charge c from such a module.

Now suppose that $f \in \mathcal{C}^\infty(S^1)$ is a smooth function with finite Fourier decomposition (which one should view as being the vector field $f(e^{i\theta})\frac{\partial}{\partial\theta}$ and let

$$f_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} f(\theta)$$

be the n-th Fourier coefficient. We then define

$$T_{(c,h)}(f) = \sum_n f_n L_n$$

which is an unbounded operator on $H_{c,h}$ with dense domain $L(c,h)$. This is really just the operator by which $-if(e^{i\theta})\frac{\partial}{\partial\theta} \in \mathrm{Vir}$ acts on $H(c,h)$ (note the extra factor of $-i$, which makes the operator self-adjoint instead of skew-adjoint, see remark 1).

Morally, we want to view $T_{(c,h)}$ as an operator-valued distribution, given by

$$T_{(c,h)}(z) = \sum_n L_n z^{-n-1}$$

on the circle. Then (morally), $T_{(c,h)}(f)$ is given by the contour integral

$$T_{(c,h)}(f) = \frac{1}{2\pi i}\int_{S^1} T_{(c,h)}(z) f(z)\cdot z dz$$

In any case, if f is real valued, the operator $T_{(c,h)}(f)$ will be essentially self-adjoint and will exponentiate to a unitary operator $e^{iT(f)}$ (some things to be checked here).

To construct the Virasoro net, we need a good chunk of functional analysis to (a) extend the above picture to all vector fields on the circle and (b) show that this yields a projective unitary action of the diffeomorphism group on $H_{(c,h)}$.

The idea is to first extend the above assignment $f \mapsto e^{iT(f)}$ to all smooth functions f using some norm estimates. Given a 1-parameter family of diffeomorphisms ϕ_t integrating a vector field X, we then define

$$U_{(c,h)}(\phi_t) := e^{it\cdot T(X)} \in PU(H) = U(H)/U(1)$$

Remark 3. *Because the Virasoro algebra is a nontrivial central extension of the vector fields, the above formula will only give a representation of the diffeomorphism group if we divide out the phase factors introduced by the cocycle. Hence we will find a projective unitary representation of the diffeomorphism group.*

Since the 1-parameter subgroups generated by a vector field lie dense in $\mathrm{Diff}^+(S^1)$, one gets a projective unitary representation of the diffeomorphism group, as shown in [3], theorem 4.2. The 'projectiveness' of this action is measured by the cocycle from the Virasoro algebra (which was also defined for all of $\mathrm{Vect}^{\mathbb{C}}(S^1)$, see equation 1).

Apart from all the technical difficulties extending stuff to all diffeomorphisms, there is one thing we can check: note that if we exponentiate elements of the Lie algebra, we generally end up in the universal cover of the group we are working with. A similar thing happens here: we end up in the universal cover of $\mathrm{Diff}^+(S^1)$, which is nontrivial since $\mathrm{Diff}^+(S^1)$ contains the rotation group $U(1)$. The universal cover is

$$\widetilde{\mathrm{Diff}^+(S^1)} = \{\phi : \mathbb{R} \xrightarrow{\sim} \mathbb{R} : \phi(x+2\pi) = \phi(x)\}$$

The map $\tilde{\phi}(e^{it}) = e^{i\phi(t)}$ gives a diffeomorphism $S^1 \to S^1$, and we get a quotient map $\widetilde{\mathrm{Diff}^+(S^1)} \to \mathrm{Diff}^+(S^1)$ whose kernel are precisely the translations by a multiple of 2π.

We have to check that we really get a projective action of $\mathrm{Diff}^+(S^1)$, not $\widetilde{\mathrm{Diff}^+(S^1)}$. To check this, let L_0 be the vector field generating the rotations. We have to check that $t \mapsto e^{it\cdot T(L_0)}$ acts by a scalar if $t \in 2\pi\mathbb{Z}$ (since actions of scalars are precisely divided out if we consider projective actions). But this follows from the fact that the eigenvalues of L_0 lie in $h + \mathbb{Z}_{\geq 0}$. Indeed, note that on any eigenvector v of L_0 with eigenvalue $h + N$ (these span the dense domain $L(c,h)$), we have

$$e^{2\pi i\cdot T(L_0)}v = e^{2\pi i\cdot(h+N)}v = e^{2\pi i\cdot h}v$$

which is independent of N. Hence $e^{2\pi i\cdot T(L_0)}$ acts by a scalar.

Remark 4. *We have already seen that Vir has a subalgebra spanned by L_{-1}, L_0, L_1, whose real part integrates to a cover of the Möbius group $PSL(2,\mathbb{R})$. Since the cocycle vanishes on this subalgebra, the above formulas actually give rise to a unitary action of the universal cover of the Möbius group.*

To see that we get a unitary representation of the Möbius group itself, we have to show again that 2π-rotations give the identity (but now in $U(H)$ instead of $PU(H)$). This forces $h \in \mathbb{Z}_{\geq 0}$.

Remark 5. *We have worked with a projective action of* $\mathrm{Diff}^+(S^1)$ *without using that we had an explicit central extension acting on the infinitesimal level. In fact, it turns out that the projective action of* $\mathrm{Diff}^+(S^1)$ *can be extended to an actual unitary action of a central extension of* $\mathrm{Diff}^+(S^1)$*. The cocycle describing this central extension is roughly the exponentiated version of the one for vector fields 1.*

1.4 The Virasoro net

After the tour de force of constructing a projective representation of $\mathrm{Diff}^+(S^1)$, one can easily define a local net by

Definition 3. *The Virsoro net is the functor*

$$A_{\mathrm{Vir},c} : I \mapsto \{U_{(c,0)}(\phi) \big| \phi \in \mathrm{Diff}^+(I) \subseteq \mathrm{Diff}^+(S^1)\}''$$

where $\mathrm{Diff}^+(I)$ *is the subset of diffeos of the circle that are the identity outside the interval* I*. It acts on the Hilbert space* $H(c,0)$ *with vacuum* $\Omega = v_0$ *the lowest weight vector.*

It will be important for the properties of a local net that we set $h = 0$. Recall that in the case $h = 0$, we have $L_{-1}v_0 = 0$.

Remark 6. *Note that the above definition makes sense, although* $U_{(c,0)}(\phi)$ *is not really an element in* $B(H)$*: indeed, if we pick any lift of this projective unitary (unique up to a phase), then* $\mathcal{A}(I)'$ *consists of all a commuting with this lift (since the lifts differ by a phase, this is independent of the chosen lift).*

Theorem 1.5. $A_{\mathrm{Vir},c}$ *is a conformal net.*

Proof. Isotony is trivial. Locality follows from remark 5: suppose ϕ only changes an interval I and ψ only changes I'. Assuming these diffeos come from flows (we can always approximate them by those), the generating vector fields have disjoint support. Consequently, the Lie algebra cocycle 1 vanishes on these vector fields.

Also globally, the smooth group cocycle (which is roughly the exponentiated version of 1) will vanish on these two diffeomorphisms. Since clearly $\phi\psi = \psi\phi$ and the cocycle vanishes, a lift of these diffeos to the central extension of $\mathrm{Diff}^+(S^1)$ commutes. This implies that $U(\phi)U(\psi) = U(\psi)U(\phi)$, even if we interpret these as true unitaries, not just elements of $U(H)/U(1)$. From this one deduces that $U(\psi) \in \mathcal{A}_{\mathrm{Vir},c}(I)'$ and therefore $\mathcal{A}_{\mathrm{Vir},c}(I') \subseteq \mathcal{A}_{\mathrm{Vir},c}(I)'$.

Next is Möbius invariance. We have already discussed the action of the Möbius group on $H_{(c,0)}$ in remark 4 For the covariance property, let $\phi \in \mathrm{Diff}^+(I)$ and $g \in$ Möb, so that $g\phi g^{-1} \in \mathrm{Diff}^+(gI)$. Then

$$U(\phi)aU(\phi)^* = a \quad \Leftrightarrow \quad U(g\phi g^{-1})\Big(U(g)aU(g^*)\Big)U(g\phi g^{-1})^* = U(g)aU(g)^*$$

which shows $U(g)\mathcal{A}(I)'U(g)^* \subseteq A(gI)'$. The same argument shows the reverse inclusion. Taking another commutant shows $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.

Observe that by definition, the generator L_0 of rotations has positive energy. The invariance of the vacuum under the Möbius group follows from the fact that

$$L_{-1}v_0 = L_0v_0 = L_1v_0 = 0$$

since $h = 0$. Moreover, it is the unique such vector (all others have a nonzero eigenvalue component for L_0, so e^{iL_0} will act nontrivially).

Finally, it is easily checked that $\mathrm{Diff}^+(S^1)$ is generated by the diffeomorphisms of the intervals. It then follows that $\bigvee \mathcal{A}(I)$ contains all lifts of $U(\phi)$, for any diffeomorphism. In particular, it will contain the operators e^{itL_n} and then also their generators L_n. Then clearly v_0 is cyclic for $\bigvee \mathcal{A}(I)$. □

Finally, we note that our net is not just Möbius covariant, but even diffeomorphism covariant (hence 'conformal'): the unitary action of the Möbius group extends to a projective action of the diffeomorphism group on $H(c,0)$ (which we have already constructed). It is easily checked that

$$U(\phi)\mathcal{A}(I)U(\phi)^* = \mathcal{A}(\phi I)$$

by the same argument as for the Möbius group.

2 Conformal net associated to a lattice

The name of this section is a bit misleading: essentially we construct a conformal net from the representation theory of (central extensions of) the loop group of a torus Lie group.

Recall the setup from Peters talk: there we considered a compact, simple, simply connected Lie group G. It turns out for such a Lie group the Lie group cohomology $H^3_{Lie}(G, U(1)) \simeq \mathbb{Z}$. After transgressing over the circle, one obtains that $H^2(LG, U(1)) \simeq \mathbb{Z}$. Correspondingly, for each level k, one obtains a (smooth) $U(1)$-central extension of the loop group LG.

We have seen that the above global picture has a nice infinitesimal analogue: infinitesimally, $H^3_{Lie}(G, U(1))$ can be described in terms of invariant polynomials on the Lie algebra $\mathfrak{g}$. In particular, the central extension of the loop Lie algebra had a simple expression in terms of invariant polynomials.

Passing to the global Lie group, one always gets an extension by a smooth group, but not necessarily by $U(1)$: for this to happen, the corresponding Lie algebra cocycle will have to be integral (otherwise, one gets an extension by $\mathbb{R}//\Gamma$ where Γ is the not-so-nice period lattice of the cocycle).

Summarizing, for simple, simply connected Lie groups, one easily gets $U(1)$-extensions: one picks an invariant polynomial on $\mathfrak{g}$ and normalizes it such that the corresponding extension of LG is really by $U(1)$.

Essentially we will do the same thing for a torus Lie group $T = U(1)^{\times n}$.

2.1 Loop group of the torus and its central extension

Let $T = U(1)^{\times n}$ be a torus Lie group and $\mathfrak{t}$ its Lie algebra. The exponential map $\exp : \mathfrak{t} \to T$ gives an isomorphism $T = \mathfrak{t}/\Lambda$, where $\Lambda = \ker \exp$ is a free lattice in $\mathfrak{t}$, of full rank (i.e. it really just looks like $\mathbb{Z}^{\times n}$ inside $\mathbb{R}^{\times n}$).

Let $LT = \mathcal{C}^\infty(S^1, T)$ be the loop group of the torus. First observe that this group is not connected, since T is not simply connected: each connected component is labeled by an element of Λ, called the winding number of the loop in the torus.

As for the diffeomorphism group, we have that LT has nontrivial first homotopy group. The universal cover of LT can be described as[2]

$$\widetilde{LT} = \Big\{ f : \mathbb{R} \to \mathfrak{t} = \mathbb{R}^n : f(\theta + 1) = f(\theta) + \Delta_f \text{ with } \Delta_f \in \Lambda \Big\}$$

The number Δ_f is precisely the winding number. (Note that none of this happens for simply connected G.) For such f, $\exp(f)$ defines a loop in T (obtained by applying the exponential map pointwise). Under the exponential map, the functions constant at Λ will be send to the trivial loop. We will often use a lift of $\exp(f) \in LT$ to an element $f \in \widetilde{LT}$.

Observe that the diffeomorphisms of S^1 act on the loop group. We will be interested in those central extensions of the loop group that carry an extension of the action of the diffeomorphism group of S^1, or rather its universal cover $\widetilde{\mathrm{Diff}^+(S^1)} = \{\phi : \mathbb{R} \xrightarrow{\sim} \mathbb{R} : \phi(\theta + 1) = \phi(\theta) + 1\}$. In other words, we have a commuting diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Diff}^+(S^1)} \times \mathcal{L}T \arrow[r] \arrow[d] & \mathcal{L}T \arrow[d] \\
\mathrm{Diff}^+(S^1) \times LT \arrow[r] & LT
\end{tikzcd}$$

where $\mathcal{L}T$ is some central extension of LT.

Proposition 2.1 ([6]). *Central extensions of LT which are invariant under the action of $\mathrm{Diff}^+(S^1)$ (meaning we can extend its action as above) are classified by symmetric bilinear forms $\langle -, - \rangle$ on $\mathfrak{t}$ such that $\langle \Lambda, \Lambda \rangle \subseteq \mathbb{Z}$ and $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in \Lambda$.*

[2] In contrast to the first section, we will identify the domain S^1 of our loops with $\mathbb{R}/\mathbb{Z}$. Otherwise lots of integrals will involve copies of 2π.

Given such an inner product $\langle -, - \rangle$, we construct a cocycle giving the central extension as follows: first, pick a bilinear map $B : \Lambda \times \Lambda \to \mathbb{Z}/2$ such that

$$B(\alpha, \alpha) = \frac{\langle \alpha, \alpha \rangle}{2} \mod 2$$

The central extension will not depend on a particular choice of B. For example, if α_i are basic elements for Λ_i (which form a basis for $\mathfrak{t}$), we can take

$$B(\alpha_i, \alpha_i) = \frac{1}{2}\langle \alpha_i, \alpha_i \rangle \qquad B(\alpha_i, \alpha_{j>i}) = \langle \alpha_i, \alpha_j \rangle \qquad B(\alpha_i, \alpha_{j<i}) = 0$$

Whatever B one picks, we always have that $(-1)^{B(-,-)}$ defines a group cocycle on Λ and

$$B(\alpha, \beta) + B(\beta, \alpha) = \langle \alpha, \beta \rangle$$

For functions $f, g \in \widetilde{LT}$, let $\hat{f}, \hat{g}$ be their means over the interval $[0, 1]$, i.e. $\hat{f} = \int_0^1 d\theta f(\theta)$ (note that this makes sense since we can view f as mapping into $\mathbb{R}^n$). We then define

$$S(f, g) = \frac{1}{2} \int_0^1 d\theta \langle f', g \rangle + \frac{1}{2}\langle \Delta_f, \hat{g} \rangle + \frac{1}{2}\langle \hat{f} - f(0), \Delta_g \rangle \qquad\qquad f' = \frac{\partial f}{\partial \theta}$$

This resembles the cocycle we defined for ordinary groups, except that now we have the winding numbers entering in the last two terms. S is bilinear. We then define the cocycle

$$c(\exp(f), \exp(g)) = (-1)^{B(\Delta_f, \Delta_g)} e^{2\pi i S(f,g)} \tag{2}$$

Note that this is independent of the chosen representatives f, g: if we add to f a constant $\alpha \in \Lambda$, then Δ_f remains the same and also $S(f + \alpha, g) = S(f, g)$. Also, if we add to g a constant $\beta \in \Lambda$ the winding number of g remains the same and S changes by

$$S(f, \beta) = \frac{1}{2} \int_0^1 \langle f', \beta \rangle + \frac{1}{2}\langle \Delta_f, \beta \rangle = \langle \Delta_f, \beta \rangle \in \mathbb{Z}$$

So the cocycle is well defined (it is a group cocycle since our group is abelian and it is multiplicative in each of its entries).

Given this cocycle c, we define a central extension by $\mathcal{L}T = LT \times U(1)$, with product given by

$$\big(\exp(f), x\big)\big(\exp(g), y\big) = \Big(\exp(f + g), xy \cdot c\big(\exp(f), \exp(g)\big)\Big)$$

where $\exp(f), \exp(g) \in LT$ and $x, y \in U(1)$. We will suggestively denote this by

$$\exp(f) \cdot \exp(g) = c\big(\exp(f), \exp(g)\big) \exp(f + g) \tag{3}$$

Although the notation does not show it, $\mathcal{L}T$ depends on the chosen bilinear form $\langle -, - \rangle$.

Lemma 2.2. *The central extension $\mathcal{L}T$ carries an action of the diffeomorphism group* $\mathrm{Diff}^+(S^1)$.

Proof. We first define an action of the universal cover of the diffeomorphism group, i.e. diffeos of the real line so that $\phi(\theta + 1) = \phi(\theta) + 1$. We define

$$\phi^*(\exp(f), x) = \Big(\exp(\phi^* f), e^{2\pi i \langle a(\phi, f), \Delta_f \rangle} x\Big)$$

with

$$a(\phi, f) = \frac{1}{2} \int_0^1 d\theta \phi^* f(\theta) - f(\theta)$$

One can check that this gives an action of the universal cover of the diffeomorphism group, which respects the multiplication in $\mathcal{L}T$ (see [5]).

To check that we get an action of $\mathrm{Diff}^+(S^1)$, we have to check that a full rotation gives the same result. A rotation over $2\pi\alpha$ is given by the diffeomorphism $\phi(t) = t + \alpha$. For $\alpha = 1$, we see that $\phi^* f(\theta) = f(\theta) + \Delta_f$ which defines the same element in T upon exponentiation. For the second component, observe that

$$a(\phi, f) = \frac{1}{2}\int_0^1 d\theta f(\theta+\alpha) - f(\theta) = \frac{1}{2}\alpha\Delta_f$$

since f is periodic, except for the constant of Δ_f. Taking $\alpha = 1$, the result follows since $\langle \Delta_f, \Delta_f \rangle \in 2\mathbb{Z}$ by assumption. □

Remark 7. *This is really a right action by* $\mathrm{Diff}^+(S^1)$*. To get the left action, we define*

$$\phi_*(\exp(f), x) = (\phi^{-1})^*(\exp(f), x)$$

2.1.1 Exercise

For the locality of the net we are going to construct, we will need the following:

Lemma 2.3. *If* $\exp(f), \exp(g)$ *have disjoint support, then* $\exp(f)\exp(g) = \exp(g)\exp(f)$*, or explicitly*

$$\big(\exp(f), x\big)\big(\exp(g), y\big) = \big(\exp(g), y\big)\big(\exp(f), x\big)$$

for any $x, y \in U(1)$ *(the choice of* x *and* y *is completely irrelevant).*

Prove it. Since you have probably spent some time reading all the stuff up to now, here are some hints:

- The condition that $\exp(f)$ and $\exp(g)$ have disjoint support means that $\exp(f)(x) = 1$ whenever $\exp(g)(x) \neq 1$ (with 1 the unit in the torus group) and vice versa. If $f' \neq 0$, this means that g can only take values in $\Lambda = \exp^{-1}(1)$.
- The main problem is in computing the integral $\int \langle f', g \rangle$. The integrand is only nonzero on those intervals I in $[0,1]$ where g is contant with value in Λ. Also observe that on the boundary of the interval (where g starts to change, if you pick the interval large enough), f necessarily lies in Λ.
- Use integration by parts.

2.2 Representations of $\mathcal{L}T$

Having found the central extension of LT, we consider its irreducible unitary representations with positive energy. The idea will be to first look at the connected component of the identity, where the representation theory is particularly easy.

Let LT^0 be the connected component of the constant loop at the unit element. It consists of those loops with winding number equal to 0 and can thus be identified with $T \oplus V$, where V is space of functions

$$f : \mathbb{R} \to \mathbb{R}^n \qquad f(x+1) = f(x) \qquad \hat{f} = 0$$

The copy of the torus consists of the constant functions, which measure the mean values on the interval $[0,1]$. Note that V is a real vector space. S vanishes on constant functions, so the cocycle 2 vanishes over the copy of T. We find that $mcLT^0$ is isomorphic to $T \oplus \tilde{V}$, where $\tilde{V}$ is the central extension of the group V defined by the cocycle 2.

It turns out that $\tilde{V}$ has a very simple representation theory:

Theorem 2.4 ([5], 9.5.10)**.** *The group* $\tilde{V}$ *has a unique unitary positive energy representation for which the central element acts nontrivially. It is denoted* $S(A)$*.*

Remark 8. *The assumption of positive energy means the following: recall that we had an action of the diffeomorphism group on* $\mathcal{L}T$ *by automorphism, which restricts to an action on* $\tilde{V}$*. In particular, the rotations act on* $\tilde{V}$*, so one can form the semidirect product* $\tilde{V} \rtimes U(1)$*. Positive energy means that the representation of* $\tilde{V}$ *extends to a representation of* $\tilde{V} \rtimes U(1)$*, where the generator of rotations is a positive operator.*

Remark 9. *Here is (very roughly) how it is constructed: V is a vector space, and the cocycle S is skew-symmetric on V. One picks a complex structure J compatible with S, i.e. $J^2 = -1$ and $S(J-,-)$ gives a real metric on V. Now complexify V and let A be the $+i$-eigenspace of J. Then A carries an Hermitean metric given by $\langle x, y\rangle = -2iS(\overline{x}, y)$, where we take the complex conjugate of x with respect to the complex structure J.*

We then take the symmetric algebra $S(A)$, which carries an inner product induced by the one on A. $\hat{S}(A)$ is the completion with respect to this inner product. It carries some nontrivial action of $\tilde{V}$. For more details, check the book by Pressley and Segal [5].

Assuming this, the following result follows immediately

Proposition 2.5. *The irreducible unitary representations of $\mathcal{L}T^0 = T \oplus \tilde{V}$ are given by $\hat{S}(A)_\alpha$, where $\alpha \in \mathrm{Hom}(T, U(1))$. As a space, it is given by $\hat{S}(A)$ with the action by $\tilde{V}$ as before, while the extra copy of T acts via α.*

Proof. The extra copy of T commutes with $\tilde{V}$, so to get irreducible representations, it should act by scalars. Since the action is required to be unitary, this means that it is specified by $\alpha \in \mathrm{Hom}(T, U(1))$. $\square$

Remark 10. *Such $\alpha \in \mathrm{Hom}(T, U(1))$ corresponds to an element in the dual lattice $\Lambda^* = \{\alpha : \langle\alpha, \beta\rangle \in \mathbb{Z}$ for all $\beta \in \Lambda\}$. Indeed, given such an element, we send $\exp(x)$ to $e^{2\pi i\langle x,\alpha\rangle}$. When $x \in \Lambda$ this gives 1, so it is well defined.*

Remark 11. *Actually, the lattice Λ itself also has a nice description: it is precisely the collection of Lie group homomorphisms $\mathrm{Hom}(U(1), T)$. Indeed, such a Lie group homomorphism defines a map of Lie algebras $\mathfrak{u}(1) = \mathbb{R} \xrightarrow{f} \mathfrak{t}$, which is uniquely defined by $f(1)$. We take as our model for $U(1)$ the quotient $\mathbb{R}/\mathbb{Z}$. The fact that f is the differential of a map $U(1) \to T$ then means that $f(1) \in \Lambda$, so $\mathrm{Hom}(U(1), T) \simeq \Lambda$.*

By the way, observe that a homomorphism $U(1) \to T$ has precisely the winding number it corresponds to in Λ.

Finally, we extend these irreducible unitary positive energy representations of $\mathcal{L}T^0$ to all of $\mathcal{L}T^0$.

Proposition 2.6. *The irreducible positive energy representations of $\widetilde{LT}$ are given by*

$$H_\lambda = \bigoplus_{\alpha\in\lambda+\Lambda} S(A)_\alpha$$

An element with winding number $\Delta_f \in \Lambda$ maps $S(A)_\alpha$ into $S(A)_{\alpha+\Delta_f}$.

Proof. This is a bit more delicate, but here is the idea. By the previous remark, we have that for each winding number $\alpha \in \Lambda$, we have a particular choice of loop in T with winding number α: we pick the homomorphism $U(1) \xrightarrow{\alpha} T$ corresponding to it. Then for any element in $\mathcal{L}T^0$, we can conjugate it by $U(1) \xrightarrow{\alpha} T$. For example, we can take a constant loop with value $\exp(x) \in T$. Then one can easily compute that

$$\exp(x)\alpha = e^{2\pi i\langle x,\alpha\rangle}\alpha \exp(x)$$

in $\mathcal{L}T$ (using notation 2.1).

This means that if $\exp(x)$ acts on $S(A)_\beta$ by $e^{2\pi i\langle x,\beta\rangle}$, then it acts on $\alpha \cdot S(A)_\beta$ (i.e. whatever we get by applying the loop α to vectors in $S(A)$) by $e^{2\pi i\langle x,\alpha+\beta\rangle}$.

So we see that the basic elements with nontrivial winding number change the element in Λ^* by which the constant loops act, essentially by adding the winding number. Then we obtain an irreducible module by summing over all $\alpha \in \lambda + \Lambda$, as in the proposition. $\square$

We thus see that the number of irreps of $\mathcal{L}T$ is equal to $|\Lambda^*/\Lambda|$, the cardinality of the dual lattice divided by the lattice. This is always finite, so there is a finite number of irreducible positive energy representations of $\mathcal{L}T$.

It turns out that we have a projective action U of the diffeomorphism group $\mathrm{Diff}^+(S^1)$ on each H_λ (which gives an actual unitary representation for Möbius transformations), with the property that

$$\phi_*(\exp(f), x) = U(\phi)(\exp(f), x)U(\phi)^*$$

In the end we can conclude:

Theorem 2.7. *Let all elements of $\mathcal{L}T$ act on $H_0 = \bigoplus_{\alpha \in \Lambda} S(A)_\alpha$. Then we define a net $\mathcal{A}$ by*

$$\mathcal{A}(I) = \{(\exp(f), x) \in \mathcal{L}T : \mathrm{supp}(\exp(f)) \subseteq I\}''$$

This defines a conformal net, with vacuum vector the element $1 \in S(A)_0$ (which was a tensor algebra).

The locality follows from the exercise, since $\exp(f)\exp(g) = \exp(g)\exp(f)$ if $\exp(f)$ and $\exp(g)$ have disjoint support.

References

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