Representation theory for conformal nets.

Today:

- Reps and the statistical dimension.

Representation up until now: defining / vacuum representation denote by π_0 / $\mathcal{H}_0$

Now: change Hilbert space, or: $\rho: \mathcal{A}(I) \to B(\mathcal{H}_\rho)$

Definition A representation ρ of the conformal net $\mathcal{A}$ on the separable Hilbert space $\mathcal{H}_\rho$ is a (consistent) family

$$\rho = \{\rho_I\}_{I \subset S^1} \qquad (I \text{ intervals!})$$

of ~~representations~~ *-representations of the local algebras $\mathcal{A}(I)$, satisfying:

i) consistence if $I \subseteq J$, then $\rho_J \restriction_{\mathcal{A}(I)} = \rho_I$

ii) Moebius covariance

$\exists$ unitary representation π^M of $\widetilde{PSL_2(\mathbb{R})}$ on $\mathcal{H}_\rho$ (i.e. projective of $PSL_2(\mathbb{R})$) such that

$$\mathrm{ad}(\pi^M(\tilde{g}))\, \rho_I(A) = \rho_{g\cdot I}(\pi_0^M(g)\, A\, \pi_0^M(g)^*)$$

$$= \rho_{g\cdot I}(\mathrm{ad}\, \pi_0^M(g)\, A)$$

(for p the canonical projection $\widetilde{PSL(2,\mathbb{R})} \to PSL(2,\mathbb{R})$, $g = p(\tilde{g})$)

iii) Positive energy / spectrum condition
The spectrum of the infinitesimal generator of rotations on $\mathcal{H}_\rho$ is positive.

First result ~~when~~ for these reps: let I, J be intervals $\subseteq S^1$.
Lemma 1 Let ρ be a rep of $\mathcal{A}$. If $I \subseteq J'$ then

$$\boxed{\rho_I(\mathcal{A}(I)) \subseteq \rho_J(\mathcal{A}(J))'}$$

proof Let $I \Subset J'$. Then $\exists$ an interval K, $I \cup J \subseteq K$ such that, using consistence
for $A \in \mathcal{A}(I)$, $B \in \mathcal{A}(J)$

$$\rho_I(A)\rho_J(B) = \rho_K(A)\rho_K(B)$$

$$= \rho_K(AB)$$

$$= \rho_K(BA) \text{ by Haag duality}$$

$$= \rho_K(B)\rho_K(A)$$
$$= \rho_J(B)\rho_J(A)$$

so $\rho_I(\mathcal{A}(I)) \subseteq \rho_J(\mathcal{A}(J))'$ for all $I \Subset J'$

If $I = J'$: $\rho_{J'}(\mathcal{A}(J')) = \bigvee_{I \Subset J'} \rho_I(\mathcal{A}(I)) \subseteq \rho_J(\mathcal{A}(J))'$

On to the def. of irreducible reps ~~###~~ and sectors

Definition A repres. ρ of $\mathcal{A}$ is <u>irreducible</u> if the vN algebra $\rho(\mathcal{A})' := \left(\vee \rho_I(\mathcal{A}(I))\right)'$ is equal to $\mathbb{C} \cdot \mathbb{1}_{\mathcal{H}_\rho}$; otherwise <u>reducible</u>

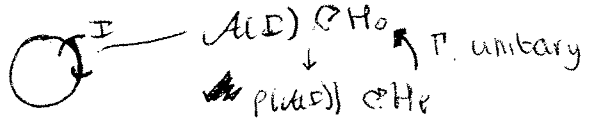
Definition Two reps ρ_1 ρ_2 are <u>unitarily equivalent</u> if $\exists$ a unitary operator $U : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$ such that

$$\rho_1(\cdot) = \mathrm{ad}(U)\, \rho_2(\cdot)$$

An equivalence class of representations of the conformal net $\mathcal{A}$ is called a <u>sector</u>. If ρ is a rep of $\mathcal{A}$,

denote sector by $[\rho]$.

Lemma 2 Any rep ρ of $\mathcal{A}$ is <u>locally</u> unitarily equivalent to the ~~identity~~ vacuum representation. So $\rho(\mathcal{A}(I)) \simeq \pi_0(\mathcal{A}(I))$



because the local algebras are all Type III_1 factors and, as we have seen before, the isomorphism class of $\mathcal{A}(I)$-modules was trivial.

Localization of a rep:

<u>Definition</u> A rep ρ (of the conformal net $\mathcal{A}_0$ on the vacuum Hilbert space $\mathcal{H}_0$) is said to be <u>localized in the interval I_0</u>, if, on the complement I_0' of I_0

$$\rho_{I_0'} = \pi_{0,I_0'} .$$

(mind: equal, not up to unitaries)

<u>Lemma 3</u> (more: result) If $[\pi]$ is a sector, then $\forall\, I_0 \subset S^1$ $\exists$ rep $\rho \in [\pi]$ localized in I_0.

<u>proof</u> By Lemma 2, $\pi_{I_0'} \cong \pi_{0\,I_0'}$, via T

So precisely: $\rho = \mathrm{ad}(T)\,\pi$

Let $\rho_1, \rho_2 \in [\rho]$ be localized in I_1, I_2. If I is an interval containing I_1 and I_2, then duality and the localization properties of ρ_1, ρ_2 imply that any operator $T_{\rho_1\rho_2}$ intertwining ρ_1 and ρ_2

$$\rho_1(A)\, T_{\rho_1\rho_2} = T_{\rho_1\rho_2}\, \rho_2(A) \quad \forall\, A \in \mathcal{A}(I),\ \forall I \subset S^1$$

belongs to $\mathcal{A}(I)$, that is: for J in the complement, $T_{\rho_1\rho_2}$ is in the commutant of $\mathcal{A}(J)$.

~~Examples~~

<u>Operations with reps</u>

i. Define „shifted reps" as examples of intertwiners
ii. Define „composition of representations"
  ↳ this is important.

Shifted representations

Gives explicit examples of intertwiners for representations ρ_1, ρ_2 (related by a Moebius transformation).

If ρ is a rep of $\mathcal{A}$ localized in I_0, define shifted reps ~~of~~ ρ_g localized in $g \cdot I_0$ by

$$\rho_{g,I}(A) :\!= (\mathrm{ad}\ \pi_0^M(g)) \circ \underbrace{\rho_{g^{-1}I} \circ (\mathrm{ad}\ \pi_0^M(g^{-1}))(A)}_{\text{recognize as möbius covariant action!}} \quad \forall A \in \mathcal{A}(I)$$

nothing but

$$\rho_g(A) = \pi_0^M(g)\, \pi_\rho^M(\tilde{g}^{-1})\, \rho(A)\, \pi_\rho^M(\tilde{g}^{-1})^*\, \pi_0^M(g)^*$$

Now define $T_\rho(\tilde{g}) := \pi_0^M(g)\, \pi_\rho^M(\tilde{g}^{-1})$, ~~the~~ intertwiner for these reps.

T~~he intertwi~~

These intertwiners, coming from $\widetilde{PSL(2,\mathbb{R})}$ satisfy the following cocycle identity

$$T_\rho(\tilde{g}_1 \cdot \tilde{g}_2) = \left(\mathrm{ad}(\pi_0^M(g_1))\, T_\rho(\tilde{g}_2)\right) T_\rho(\tilde{g}_1)$$

Can also work back of course: given the intertwiners / cocycles one can restore ~~the~~ a rep related to ρ via some Moebius transform.

$*$: $\rho_J(\mathcal{A}(J)) \subseteq \rho_{J'}(\mathcal{A}(J'))'$ by locality
$= \pi_{0_{J'}}(\mathcal{A}(J'))'$
$= \mathcal{A}(J')' = \mathcal{A}(J)$ by Haag duality

Composition of sectors

Lemma (small) If ρ is localized in I_0, then for any interval containing I_0, J, one has

$* \quad \rho_J(\mathcal{A}(J)) \subseteq \pi_0(\mathcal{A}(J)) = \mathcal{A}(J)$

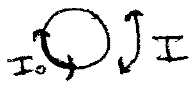
subfactor!

i.e: ρ_J is an endomorphism of $\mathcal{A}(J)$!
We know how to compose endomorphisms, we know that this corresponds to Connes fusion in terms of bimodules.
Today: more endomorphisms.

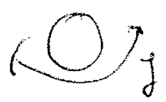
~~Given~~ **Definition** Given two sectors $[\rho_1]$, $[\rho_2]$, we define a composed sector $[\rho_1 \hat{\circ} \rho_2]$ by picking two representations $\rho_1 \in [\rho_1]$, $\rho_2 \in [\rho_2]$ localized in some common interval I_0 (I can do this by lemma!) and defining a composed rep. $\rho_1 \hat{\circ} \rho_2$ on a per-interval basis.

2 cases.

- Let I be such that ~~there~~ $\exists\, J \subset S^1$ with $I \cup I_0 \subseteq J$



then by the small lemma, we have that



ρ_1 and ρ_2 are both endomorphisms on $\mathcal{A}(J)$, which contains $\mathcal{A}(I)$ by isotony!, so

$$\boxed{(\rho_1 \hat{\circ} \rho_2)_I := \rho_{1,J} \circ \rho_{2,J} \restriction \mathcal{A}(I)}$$

Case 2 If I is such that $\overline{I \cup I_0}$ covers S^1, choose $\tilde{I}_0 \subset I_0$ such that $\tilde{I}_0 \cup I \subseteq J$ for some J (i.e.: make I_0 smaller to fit case 1). Now ~~by the~~ by the lemma 2, $\exists$ $\tilde{\rho}_1, \tilde{\rho}_2 \in [\rho_1], [\rho_2]$ localized in $\tilde{I}_0$ with intertwiners $T_{\rho_1\tilde{\rho}_1}$, $T_{\rho_2\tilde{\rho}_2}$

$$\rho_i = \mathrm{ad}(T_{\rho_i\tilde{\rho}_i})\, \tilde{\rho}_i$$

~~Lemma~~ This lemma tells you our definition is decent, i.e. it does not depend on choices made (e.g. J). Also, it is consistent

Lemma The composed representation $\rho_1 \hat{\circ} \rho_2$ is well-defined, i.e., one has that

i) the definition of $(\rho_1 \hat{\circ} \rho_2)_I$ is indep. of the particular choices made $\forall I \subset S^1$.

ii) If $I \subseteq J$, then $(\rho_1 \hat{\circ} \rho_2)_J \restriction \mathcal{A}(I) = (\rho_1 \hat{\circ} \rho_2)_I$

proof: Long ~~too~~, but does use almost everything up until now, proof of i) will be online in $\TeX$, as original paper contains typo's and can use some compacter notation.

Nice properties of $\rho_1 \hat{\circ} \rho_2$:

i) The representation $\rho_1 \hat{\circ} \rho_2$ is localized in any interval $I_1 \supset I_0$

ii) Unitary equivalence: Let $\tilde{\rho}_1, \tilde{\rho}_2$ be unitarily equiv. reps localized in a common interval I_1, then

$$\rho_1 \hat{\circ} \rho_2 \cong \tilde{\rho}_1 \hat{\circ} \tilde{\rho}_2$$

~~If ∃ interval J ∋ $I_0 \cup I_n$~~
If $\exists$ interval J containing $I_0 \cup I_1$,
then one can express the intertwiners of $\tilde{\rho}_1 \hat{\circ} \tilde{\rho}_2$
in terms of those of $\tilde{\rho}_1$ and $\tilde{\rho}_2$.

iii) If ρ_1 is localized in $I_1 \subseteq I_0$, ρ_2 in $I_2 \subset I_0$
and $I_1 \cap I_2 = \emptyset$ then
$$\rho_1 \hat{\circ} \rho_2 = \rho_2 \hat{\circ} \rho_1$$

iv) $\rho_1 \hat{\circ} \rho_2$ is again Mobius covariant.

— BREAK-?

~~BRAID STATISTICS~~

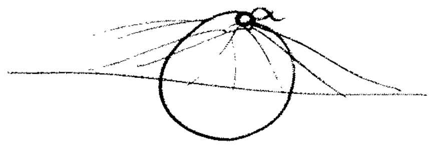
Why are we even interested in localized representations? According to Doplicher, Haag, Roberts, the physical representations are those that differ observably from the vacuum representation only in local regions. After the above composition, can see we are working towards ~~catego~~ information of the category of reps of the algebra.

Idea to keep in mind: we are looking at the analysis of representations by looking at endomorphisms.

„shifted reps" $\Rightarrow$ transportable endomorphisms.

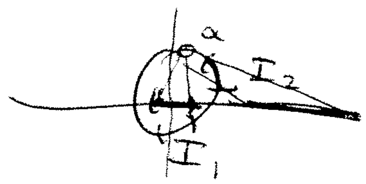
BRAID STATISTICS

Definition Let $\alpha \in S^1$ and p_α be



- point at infinity.

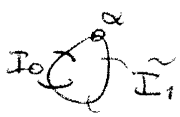
If I_1, I_2 are 2 intervals on S^1 s.t.
$I_1 \cap I_2 = \emptyset$ and which are mapped
by p_α to bounded intervals, can define
$I_1 <_\alpha I_2$ on the left / right



$I_1 <_\alpha I_2$

$I_{1\mathbb{R}} := p_\alpha(I_1)$.

Definition Let ρ_1, ρ_2 be 2 representations localized in I_0.
Pick a rep $\tilde{\rho}_1 \in [\rho_1]$ loc. in $\tilde{I}_1$ s.t. $I_0 \cap \tilde{I}_1 = \emptyset$,
$\alpha \notin \tilde{I}_1$, $\tilde{I}_{1\mathbb{R}} >_\alpha I_0$



And $\hat{\rho}_1$, same but $\hat{I}_1 <_\alpha I_0$
Let $\tilde{J}$ / $\hat{J}$ be an interval containing $I_0 \cup \tilde{I}_1$ / $I_0 \cup \hat{I}_1$

Let $T_{\rho_1\tilde{\rho}_1}$, $T_{\rho_1\hat{\rho}_1}$ be intertwiners
Define

$$\varepsilon^+_{\rho_1 \circ \rho_2} := T_{\rho_1\tilde{\rho}_1}\, \rho_{2,J}\,(T^*_{\rho_1\tilde{\rho}_1}) \quad \in \mathcal{A}(J)$$

$$\varepsilon^-_{\rho_1 \circ \rho_2} := T_{\rho_1\hat{\rho}_1}\, \rho_{2,\tilde{J}}\,(T_{\rho_1\hat{\rho}_1}) \quad \in \mathcal{A}(\tilde{J}).$$

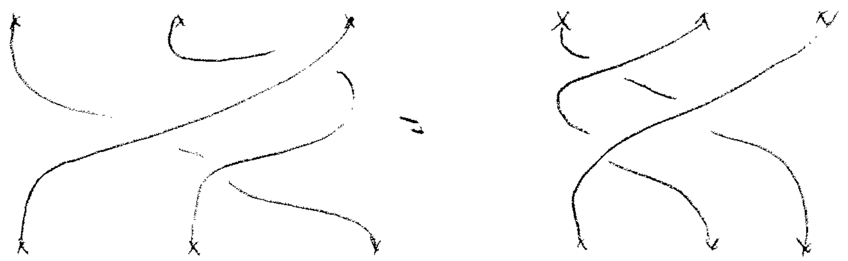
These $\varepsilon^{\pm}_{\rho_1 \hat{\circ} \rho_2}$ are the <u>intertwiners</u> for $\rho_1 \hat{\circ} \rho_2$ and $\rho_2 \hat{\circ} \rho_1$!
In addition, they satisfy $\varepsilon^{+}_{\rho_1 \hat{\circ} \rho_2} \bullet \varepsilon^{-}_{\rho_1 \hat{\circ} \rho_2} = \mathbb{1}_{\mathcal{H}_0}$

<u>definition</u> Braid group B_n. n strands with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

i. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $\quad i = 1, \dots, n-2$
ii. $\sigma_i \sigma_j = \sigma_j \sigma_i$ $\quad |i-j| > 1$ $\quad i,j = 1, \dots, n-1$

(anyons)

Rule i graphically:



<u>Action of the braid group on our representations</u>

<u>Define</u> $\rho^{\hat{\circ} i} := \rho \hat{\circ} \rho \hat{\circ} \dots \hat{\circ} \rho$ (i times)
for ρ a rep ~~with~~ localized in some interval.

Now
$\pi^{\rho}_{n}(\sigma_i) := \rho^{\hat{\circ} i-1}(\varepsilon^{+}_{\rho \hat{\circ} \rho})$

<u>Theorem</u>
- $\pi^{\rho}_{n} : B_n \to \rho^{\hat{\circ} n}(\mathcal{A})'$ extends to a unitary rep. of the braid group B_n.

- This rep. only depends only on the sector of ρ!

Left inverse

To introduce this, must first define $\bar{\mathcal{A}}^d_\alpha$

$\mathcal{A}_\alpha := \{ \mathcal{A}_\alpha(I) \mid I \subset \mathbb{R} \text{ bounded} \}$, α pt at infty

$\downarrow$

$\mathcal{A}^d_\alpha := \{ \mathcal{A}^d_\alpha(I) := \mathcal{A}_\alpha(I'^{\mathbb{R}})' \mid I \subset \mathbb{R} \text{ bounded} \}$

$\downarrow$

$\overline{\mathcal{A}^d_\alpha} = \overset{C^*}{\text{inductive}}$ limit! Adds quasi-local observables.

Definition ~~Inductive system~~
Because of isotony ($\mathcal{O} \subseteq \hat{\mathcal{O}} \Rightarrow \exists\, i: \mathcal{A}(\mathcal{O}) \hookrightarrow \mathcal{A}(\hat{\mathcal{O}})$)
our nets are examples of inductive systems.

Def Let Λ be a directed set. So, Λ has a preorder $\leq$ s.t. for each λ_1, λ_2 in Λ $\exists\, \lambda \in \Lambda$ with $\lambda_i \leq \lambda$ $i = 1, 2$.

An inductive system of C^*-algebras is a collection $\{ (\mathcal{A}_{\lambda_1}, i_{\lambda_1 \lambda_2}) : \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \leq \lambda_2 \}$ where $i_{\lambda_1 \lambda_2}$ is a $*$-homomorphism from the C^*-algebra $\mathcal{A}_{\lambda_1}$ to $\mathcal{A}_{\lambda_2}$ s.t.

$i_{\lambda_2 \lambda_3} \circ i_{\lambda_1 \lambda_2} = i_{\lambda_1 \lambda_3} \quad \forall\, \lambda_1 \leq \lambda_2 \leq \lambda_3$.

Def With an inductive system, can take <u>inductive limit</u> in the category of

C^*-algebras.

In essence: consider a subalgebra A of $\prod_\lambda \mathcal{A}_\lambda$ subject to the condition that $\exists\ \lambda_0$ s.t. $\iota_{\lambda_1,\lambda_2}(\underset{B_{\lambda_1}}{\cancel{A_{\lambda_1}}}) = B_{\lambda_2}$ $\forall\ \lambda_2 \geq \lambda_1 \geq \lambda_1$.

This algebra can be endowed with a seminorm, mod out by kernel of seminorms, then taking completion $\rightarrow$ <u>inductive limit</u>.

In AQFT: Take a union of $\mathcal{A}(I)$, then make it into a vN-algebra again.
Or: $\overline{\bigvee \mathcal{A}(I)}^{\,norm}$.

So, ~~self adjo~~ <u>left-inverse</u>

Def A positive linear map $\varphi : \bar{\mathcal{A}}^d_\alpha \rightarrow B(\mathcal{H})$ is called a left inverse for ρ on $\bar{\mathcal{A}}^d_\alpha$ if

$\varphi(A\rho(B)) = \varphi(A)B \qquad \forall\ A, B \in \mathcal{A}^d_\alpha$

$\varphi(\mathbb{1}_\mathcal{H}) = \mathbb{1}_\mathcal{H}$

<u>Big lemma</u>

Let ρ be ~~morally irreducible~~ localized in I_0 and irreducible.

Let φ be the left inverse for ρ on $\bar{\mathcal{A}}^d_\alpha$.

Then

- $\varphi(\varepsilon^+_{\rho\rho}) = \lambda \cdot \mathbb{1}_\mathcal{H}$ for some $\lambda \in \mathbb{C}$
- $\varphi(\varepsilon^-_{\rho\rho}) = \bar{\lambda}\ \mathbb{1}_\mathcal{H}$

If $\exists$ this left inverse, ρ has finite statistical Dimension. $\boxed{d(\rho) := \lambda^{-1}}$