This document contains the definition of the composition of two localized representations. In addition to the definition, there is a proof to show it is well-defined.

DEFINITION 0.1. Let ρ_1 and ρ_2 be two representations of a conformal net $\mathcal{A}$ on the circle and let them both be localized in the interval I_0. We will be looking at their composition, $\rho_1 \hat{\circ} \rho_2$. This is defined again as a family $\{\rho_1 \hat{\circ} \rho_2\}_I$, $I \subset S^1$, of representations of the local algebras $\mathcal{A}(I)$. (In the following, $i = 1, 2$)

- For I such that there exists an interval J containing both I and I_0, we use the fact that by Haag duality the two representations $(\rho_i)_J \in \mathcal{A}(J)$. Hence, we can define
$$(\rho_1 \hat{\circ} \rho_2)_I(A) := ((\rho_1)_J \circ (\rho_2)_J)(A) \ \forall A \in \mathcal{A}(I).$$
- For the second case, let I and I_0 be such that their closure covers S^1. Define $\tilde{I_0} \subset I_0$ to be such that we are in case 1, i.e, there exists an interval J containing both I and $\tilde{I_0}$ and take $\tilde{\rho}_i$ to be the representations localized in $\tilde{I_0}$ and unitarily equivalent to ρ_i. We now define for I and $A \in \mathcal{A}(I)$:
$$(\rho_1 \hat{\circ} \rho_2)_I(A) := \mathrm{ad}(\Gamma_{\tilde{1}})\mathrm{ad}(\tilde{\rho}_{1,I_0}(\Gamma_{\tilde{2}}))\tilde{\rho_{1,J}} \circ \tilde{\rho_{2,J}}(A),$$
where we denoted with $\Gamma_{\tilde{i}}$ the intertwiner for ρ_i and $\tilde{\rho}_i$.

THEOREM 0.2. *Let ρ_1 and ρ_2 be as above. The composition $\rho_1 \hat{\circ} \rho_2$ is well-defined, i.e. it does not depend on the choices made in the definition.*

PROOF. For the first case, this is left as a homework exercise. The second part might be the hardest, but can be simplified greatly with the additivity property of conformal nets. We will give the proof without the assumption of (strong) additivity.
Let I be such that $I_0 \cup I$ lies dense in S^1. Throughout this proof, A is an element of $\mathcal{A}(I)$. After assuming case 1 to be well-defined, all that remains to check is the dependency of the choice of $\tilde{I_0}$. Let $\hat{I_0}$ denote another interval inside I s.t. $\hat{I_0} \cup I$ do not lie dense in S^1, but with $\hat{I_0} \subset \tilde{I}_0$, and let $\hat{\rho_i}$ be the two representations localized in $\hat{I_0}$ with intertwiners $\Gamma_{\hat{i}}$. We write out the two definitions of $\rho_1 \hat{\circ} \rho_2$ with the two different choices:
$$\begin{aligned}(\rho_1 \hat{\circ} \rho_2)_I(A) &= \mathrm{ad}(\Gamma_{\tilde{1}})(\mathrm{ad}(\tilde{\rho}_{1,I_0}(\Gamma_{\tilde{2}}))\tilde{\rho_{1,J}} \circ \tilde{\rho_{2,J}}(A)) \\ &= \mathrm{ad}(\rho_{1,I_0}(\Gamma_{\tilde{2}}))(\mathrm{ad}(\Gamma_{\tilde{1}})\tilde{\rho_{1,J}} \circ \tilde{\rho_{2,J}}(A)) \\ &\stackrel{?}{=} \mathrm{ad}(\Gamma_{\hat{1}})(\mathrm{ad}(\hat{\rho}_{1,I_0}(\Gamma_{\hat{2}}))\hat{\rho_{1,J}} \circ \hat{\rho_{2,J}}(A)) \\ &= \mathrm{ad}(\rho_{1,I_0}(\Gamma_{\hat{2}}))(\mathrm{ad}(\Gamma_{\hat{1}})\hat{\rho_{1,J}} \circ \hat{\rho_{2,J}}(A)).\end{aligned}$$

Hence, we have equality if and only if:

$$\widetilde{\rho_{1,J}} \circ \widetilde{\rho_{2,J}}(A) = \mathfrak{ad}(\Gamma_{\tilde{1}}\rho_{1,I_0}(\Gamma^*{}_{\tilde{2}})\rho_{1,I_0}(\Gamma_{\hat{2}})\Gamma_{\hat{1}})(\widehat{\rho_{1,J}} \circ \widehat{\rho_{2,J}})(A) \tag{0.1}$$

We shall show that equation 0.1 holds in our case. Use that we have an intertwiner between $\tilde{\rho_2}$ and $\hat{\rho}_i$, which is localized in $\tilde{I}_0$:

$$\Gamma^*_{i\tilde{i}}\Gamma_{i\hat{i}} =: \Gamma_{\tilde{i}\hat{i}}.$$

Now since $\tilde{I}_0 \subset I_0$ and $\tilde{I}_0 \subset J$, it follows from the localization property of $\tilde{\rho}_1$ that $\tilde{\rho}_{1,J}(\Gamma_{\tilde{2}\hat{2}}) = \widetilde{\rho_{1,I_0}}(\Gamma_{\tilde{2}\hat{2}})$. Using this intertwiner for equation 0.1 we get (after moving $\Gamma_{\tilde{1}}$ through):

$$\begin{aligned}
\mathfrak{ad}(\tilde{\rho}_{1,J}(\Gamma_{\tilde{2}\hat{2}})\Gamma_{\tilde{1}\hat{1}})\widehat{\rho_{1,J}} \circ \widehat{\rho_{2,J}}(A) &= \mathfrak{ad}(\widetilde{\rho_{1,J}})(\widetilde{\rho_{1,J}} \circ \hat{\rho}_{2,J})(A) \\
&= \tilde{\rho}_{1,J}(\mathfrak{ad}(\Gamma_{\tilde{2}\hat{2}})\hat{\rho}_{2,J}(A)) \\
&= \widetilde{\rho_{1,J}}(\tilde{\rho}_{2,J}(A)) \\
&= \widetilde{\rho_{1,J}} \circ \widetilde{\rho_{2,J}}(A)
\end{aligned}$$

So indeed, the composition of $\hat{\rho}_i$ and the composition of $\tilde{\rho}_i$ agree.
So far we have looked at the case of shrinking the interval after the choice of $\tilde{I}_0$. However, left is to prove that the definition is independent of initial choice of $\tilde{I}_0$. I.e: looking at the definition

$$(\rho_1 \hat{\circ} \rho_2)_I(A) = \mathfrak{ad}(\Gamma_{\tilde{1}}\tilde{\rho}_{1,I_0}(\Gamma_{\tilde{2}}))(A)$$

should not depend on the choice of $\tilde{I}_0$ (with $\tilde{I}_0 \cap I = \emptyset$ in this case, by using our proof above!). Again, let $\hat{I}_0$ denote another choice of interval satisfying all the requirements of $\tilde{I}_0$. THere exists an interval J_0 such that $\tilde{I}_0 \cup \hat{I}_0 \subset J_0$ while $J_0 \cap I = \emptyset$. The condition for the agreement of the composition for both $\hat{I}_0$ and $\tilde{I}_0$ is

$$\mathfrak{ad}(\Gamma^*{}_{\hat{1}}\rho_1(\Gamma^*{}_{\hat{2}})\rho_1(\Gamma_{\tilde{2}})\Gamma_{\tilde{1}})(A) \stackrel{?}{=} \pi_0(A), \tag{0.2}$$

where π_0 denotes the vacuum representation. Again, by using the intertwiners $\Gamma_{\tilde{i}\hat{i}}$ we get:

$$\begin{aligned}
\Gamma^*{}_{\hat{1}}\rho_1(\Gamma^*{}_{\hat{2}})\rho_1(\Gamma_{\tilde{2}})\Gamma_{\tilde{1}} &= \Gamma^*{}_{\hat{1}}\rho_1(\Gamma^*{}_{\hat{2}}\Gamma_{\tilde{2}})\Gamma_{\tilde{1}} \\
&= \Gamma^*{}_{\hat{1}}\rho_1(\Gamma_{\hat{2}\tilde{2}})\Gamma_{\tilde{1}} \\
&= \hat{\rho}_1(\Gamma_{\hat{2}\tilde{2}})\Gamma^*{}_{\hat{1}}\Gamma_{\tilde{1}} \\
&= \hat{\rho}_1(\Gamma_{\hat{2}\tilde{2}})\Gamma_{\hat{1}\tilde{1}}
\end{aligned}$$

Now we use this together with the localization property of ρ: we know our intertwiners $\Gamma_{\hat{i}\tilde{i}}$ belong to $\mathcal{A}(J_0)$, giving also that $\hat{\rho}(\Gamma_{\hat{2}\tilde{2}})\Gamma_{\hat{1}\tilde{1}}$ lies inside $\mathcal{A}(J_0)$. Thus, this is in the commutant of $\mathcal{I}$ and taking the ad-operation gives us the identity (that is: the vacuum representation) back. Hence, we fulfill condition 0.2 and we are done. $\square$

Exercise 0.1. (due date: May 24th 2013) In this exercise you shall make the proof of the above theorem a lot more easy, by proving the independence of choices for case 1 and then making an elegant argument on case 2 using additivity.

- First, prove the first case (the simple case of compositions). That is, for the case that there exists an interval J such that I and I_0 are contained in J (case 1), you want to check it is independent of the choice of J. You can make a distinction between $\mathrm{I} \cap \mathrm{I}_0 \neq (=) \varnothing$.
- Last week André sketched that the second case can be simplified greatly if one uses additivity. Work this out. If really needed, you can use strong additivity, but explicitly note in which case(s) you have to use it.