String diagrams and algebra
Hand-in exercise 4 (for 21-11)

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Warm-up: vector spaces and tensor products

We start by introducing diagrammatic notation that will be used in this exercise, where we denote maps by little strings. Let V be a vector space over $\mathbf{C}$. Rather than depicting V itself we draw the identity map $\mathbf{1}\colon V \longrightarrow V$, which simply looks like in Figure 1 on page 4 below.

Since we are in the realm of linear algebra, all our objects (vector spaces, associative algebras, and so on) come equipped with a linear structure. Accordingly, all maps that we write down should be compatible with that structure: they should be (multi)linear. Recall the axioms for the *tensor product* $U \otimes V$ of two vector spaces over $\mathbf{C}$: for all $u, u' \in U$, all $v, v' \in V$ and all $\lambda \in \mathbf{C}$ we have

$$(u + u') \otimes v = u \otimes v + u' \otimes v \ , \quad u \otimes (v + v') = u \otimes v + u \otimes v' \ , \quad (\lambda u) \otimes v = u \otimes (\lambda v) = \lambda(u \otimes v) \ .$$

i) Use these axioms to check that any map $U \otimes V \longrightarrow W$ is bilinear, and reversely, that any bilinear map $U \times V \longrightarrow W$ yields a map $U \otimes V \longrightarrow W$.

Thus, we shall need a graphical notation shown in Figure 2 for the tensor product of two vector spaces, or rather, of the identity map $\mathbf{1}_{U\otimes V} = \mathbf{1}_U \otimes \mathbf{1}_V$ on the tensor product. Of course there is also a 'braiding' or permutation operator interchanging the two elements, which we depict as in Figure 3. Sometimes we also want to depict the (identity map on the) boring vector space $\mathbf{C}$ itself. That's easy: we do not draw anything, which is compatible with the fact that $\mathbf{C} \otimes V \cong V \cong V \otimes \mathbf{C}$ are essentially the same.

Algebras and Lie algebras

Let's go forth and multiply. To define an *(associative) algebra* we start with a vector space A with a map $m\colon A \otimes A \longrightarrow A$ (note that in this way we've already said that m is bilinear). We draw m as shown in Figure 4. Actually, since the shape of the diagram in the middle is very distinctive, it won't cause any confusion to simply depict m as in Figure 5. Similarly we can depict the unit map $\eta\colon \mathbf{C} \longrightarrow A$ (sending scalars to the corresponding multiples of the unit element $1_A \in A$) in terms of a string diagram: see Figure 6 and its shorthand in Figure 7.

ii) Translate the commutative diagrams in Fig. 1.1 of [Maj] expressing the axioms for associative algebras into string diagrams.

iii) Since any algebra is also a vector space, we know that the tensor product $A \otimes B$ of two algebras is also a vector space. To turn it into an algebra we have to define a multiplication and unit map for $A \otimes B$. Translate the usual definitions $\eta_{A\otimes B} \coloneqq \eta_A \otimes \eta_B$ and $(a \otimes b) \cdot (a' \otimes b') \coloneqq (aa') \otimes (bb')$ into string diagrams, and check that this turns $A \otimes B$ into an associative algebra.

iv) A *commutative* algebra satisfies $ab = ba$ for all $a, b \in A$. Use the maps that we have introduced so far to write down a commutative diagrams expressing this axiom, and give the corresponding string diagram.

v) For comparison consider a Lie algebra $\mathfrak{g}$. As the bracket $[\cdot, \cdot]$ also is a map $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$, we can depict it with the same string diagram as for m. (Lie algebras do not come with a multiplication, so there should be no confusion in this case.) Draw string diagrams expressing the axioms for Lie algebras.

Coalgebras and bialgebras

Reversing the direction of the arrows in the commutative diagrams stating the axioms for an associative algebra, we obtain the axioms for a coalgebra C. These involve a comultiplication $\Delta\colon C \longrightarrow C \otimes C$ and a counit $\varepsilon\colon C \longrightarrow \mathbf{C}$, which we can draw as shown in Figure 8.

vi) Convert the commutative diagrams in Fig. 1.2 of [Maj] into string diagrams. What does it mean for Δ to be cocommutative?

vii) The three upper commutative diagrams in Fig. 1.3 of [Maj] do not involve the antipode, and are really part of the definition of a bialgebra. Draw those axioms in terms of string diagrams.

Our favourite example of a bialgebra is of course the Yang-Baxter algebra from Section 2.1.3 of [GRS]. Recall that by definition, a *Yang-Baxter algebra* $\mathscr{A}$ consists of

- A family of invertible matrices $R(u) \in \mathrm{Aut}(\mathbf{C}^n \otimes \mathbf{C}^n)$ parametrized by a *spectral parameter* $u \in \mathbf{C}$ (here $\mathbf{C}^n$ is the *auxiliary space*).
- An associative algebra T with a unit element $\mathbf{1}$ and a family of n^2 generators $\{T_i^j(u)\}_{1\leq i,j\leq n}$ parametrized by $u \in \mathbf{C}$, subject to the following relation: two generators with different spectral parameter can be commuted according to the Yang-Baxter equation (2.24):

$$\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} R_{j_1 j_2}^{k_1 k_2}(u-v)\, T_{i_1}^{j_1}(u)\, T_{i_2}^{j_2}(v) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} T_{j_2}^{k_2}(v)\, T_{j_1}^{k_1}(u)\, R_{i_1 i_2}^{j_1 j_2}(u-v) \ .$$

By definition, then, the Yang-Baxter is an associative algebra. We have also seen that it has a comultiplication $\Delta\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$ which is determined by $T_i^j(u) \longmapsto \sum_{k=1}^{n} T_i^k(u) \otimes T_k^j(u)$.

viii) Show that this comultiplication is coassociative, and give the formula for the counit. Check that this turns the Yang-Baxter algebra into a bialgebra by verifying the axioms from part (vii).

In the seminar we have already encountered other examples of bialgebras, but those actually have more structure, which we turn to next.

Hopf algebras

To state the axioms for a Hopf algebra H we further need the antipode $S\colon H \longrightarrow H$, which we draw as in Figure 9.

ix) Translate the remaining commutative diagram in Fig. 1.3 of [Maj] into string diagrams.

This final axiom has some nice direct consequences, given in Proposition 1.4 of [Maj].

x) Convert that proposition and its proof into string diagrams.

We conclude this exercise with two examples that we have encountered in this week's presentation. For the first class of examples, let $\mathfrak{g}$ be any Lie algebra, and consider its universal enveloping algebra $U(\mathfrak{g})$, which can be obtained from the tensor algebra

$$T(\mathfrak{g}) = \mathbf{C} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \oplus \cdots$$

by forcing the bracket of $\mathfrak{g}$ inside $U(\mathfrak{g})$ to coincide with the commutator bracket. Again, by construction this is an associative algebra.

For $X \in \mathfrak{g}$ we set

$$\Delta(X) \coloneqq X \otimes \mathbf{1} + \mathbf{1} \otimes X \ , \qquad \varepsilon(X) \coloneqq 0 \ , \qquad S(X) \coloneqq -X \ . \tag{1}$$

xi) Check that these definitions turn the associative algebra $U(\mathfrak{g})$ into a cocommutative Hopf algebra.

As we have seen today, if $\{X_i\}_{1\leq i\leq n}$ is a basis for $\mathfrak{g}$, then a convenient basis for $U(\mathfrak{g})$ is given by $\{X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}\}_{r_1,\cdots,r_n\in\mathbf{N}}$. This basis is known as the Poincaré-Birkhoff-Witt, or PBW, basis. The point is not that it consists of products of the X_i, but that we can use the bracket in $U(\mathfrak{g})$ to express any of its element as a linear combination of this 'lexicographically ordered' basis. For example (letting c_{ijk} denote the structure constants of $\mathfrak{g}$):

$$X_2 X_1 = X_1 X_2 - [X_1, X_2] = X_1 X_2 - \sum_n c_{12n}\, X_n \ .$$

xii) Extend (1) to the PBW basis: compute the result of applying these maps to $X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}$. Is the counit map identically equal to zero?

Finally take $\mathfrak{g} = \mathfrak{sl}_2$ and consider the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. As before this is an associative algebra by construction, now subject to the relations

$$KK^{-1} = K^{-1}K = \mathbf{1} \ , \qquad KEK^{-1} = q^2 E \ , \qquad KFK^{-1} = q^{-2} F \ , \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \ .$$

Further set

$$\begin{aligned} &\Delta(E) \coloneqq E \otimes K + \mathbf{1} \otimes E \ , \qquad \Delta(F) \coloneqq F \otimes \mathbf{1} + K^{-1} \otimes F \ , \qquad \Delta(K^{\pm 1}) \coloneqq K^{\pm 1} \otimes K^{\pm 1} \ , \\ &\varepsilon(E) \coloneqq \varepsilon(F) \coloneqq 0 \ , \qquad \varepsilon(K^{\pm 1}) \coloneqq 1 \ , \\ &S(E) \coloneqq -EK^{-1} \ , \qquad S(F) \coloneqq -KF \ , \qquad S(K^{\pm 1}) \coloneqq K^{\mp 1} \ . \end{aligned}$$

xii) These definitions turn $U_q(\mathfrak{sl}_2)$ into a Hopf algebra. Prove that they satisfy the coassociativity axiom and the Hopf algebra axiom, and that $U_q(\mathfrak{sl}_2)$ is not cocommutative.

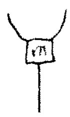
Figures

1. $\mathbb{1}_V : V \to V$

2. $U\otimes V \to U\otimes V$

3. $\tau : U\otimes V \to V\otimes U$

4. $m : a\otimes b \mapsto ab$



5. 

6. $\eta : \lambda \mapsto \lambda 1_A$

7. 

8. 

9.