In some sense, quantum groups are a generalization of Lie algebras. So let me start with Lie algebras

(here: Lie algebra = finite dim. simple Lie algebra)

<u>First example:</u> $sl(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a+d=0 \right\}$ ← I don't want to think about it that way

$sl(2) = \mathrm{span}\{H, E, F\}$ with

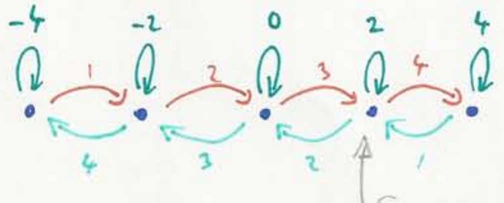
Lie bracket given by $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$

$\left(H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right.$ but I don't care $\left. \right)$

The Lie algebras are important, and it's good to understand their structure. But much more important is to understand the structure of their representations (and that's also going to be the case for quantum groups)

<u>Reps of $sl(2)$:</u>

1) Finite dimensional irred representations
for every natural number, there is exactly one irrep of that dimension



H E F

let's check the relation $[E,F] = H$ on this basis vector

$\underbrace{EF}_{6} - \underbrace{FE}_{4} = \underbrace{H}_{2}$ ✓

Another example: $sL(3)$

Before telling you what the Lie algebra is, I will tell you what its representations look like

Here when I say "this basis vector goes to that basis vector", I'm not indicating which multiple

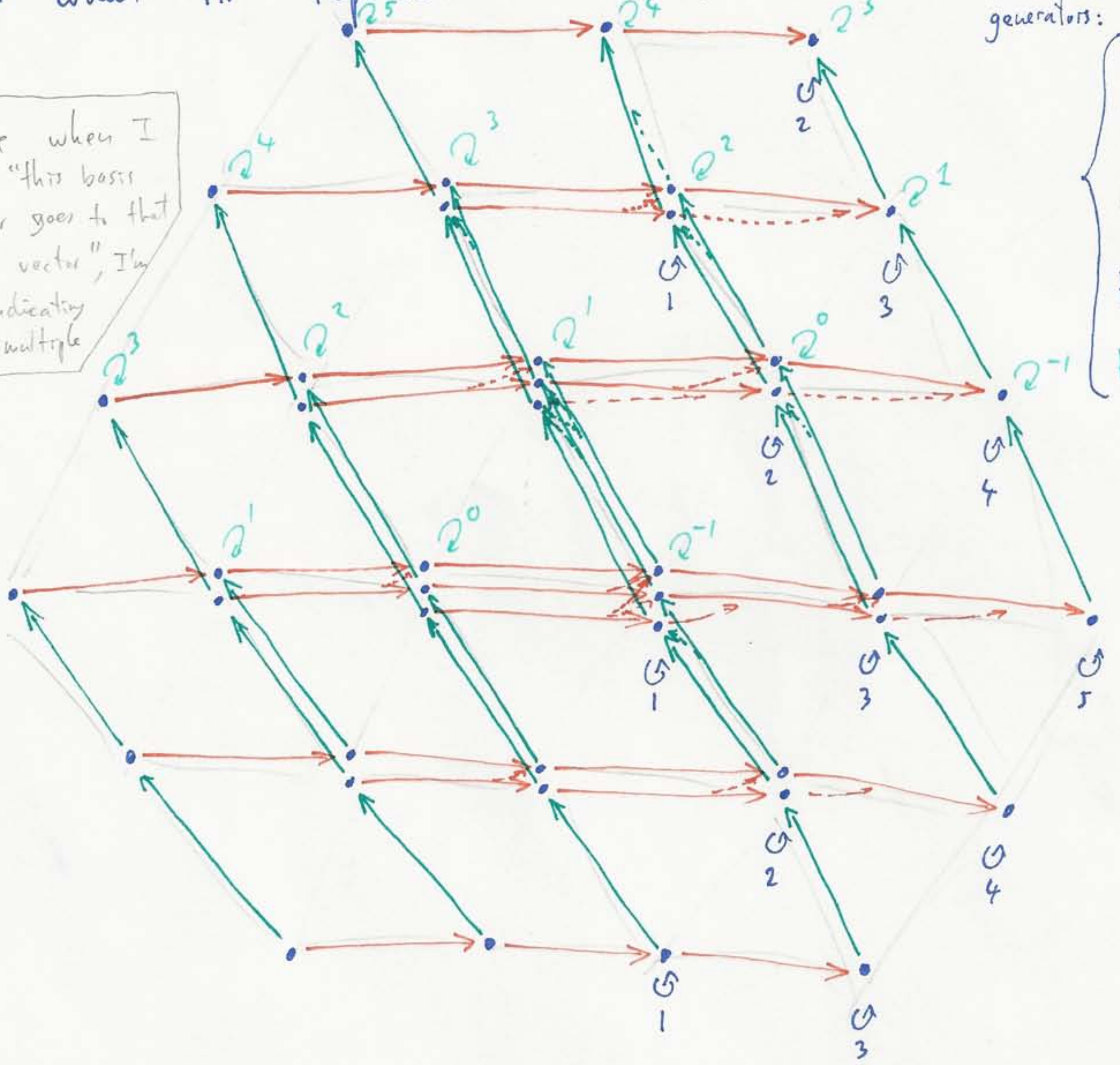
generators:

E_1 $(F_1 = E_1^*)$ (up to scalars)

E_2 $(F_2 = E_2^*)$

H_1

H_2



$E_1 = \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}$

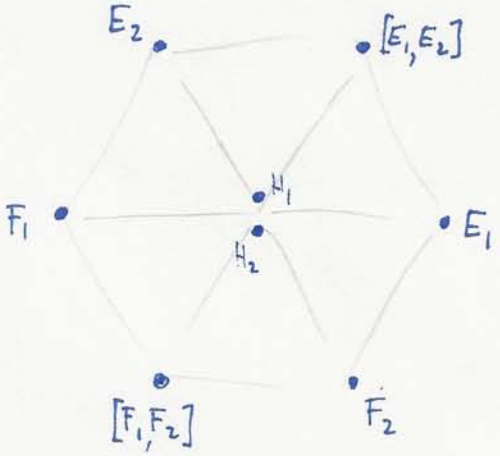
$E_2 = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}$

$H_1 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}$

$H_2 = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{pmatrix}$

(actually, I'm lying a tiny bit ...)

The Lie algebra $sL(3)$ itself is:



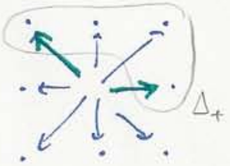
This picture  is called the root system of $\mathfrak{g}$

positive roots Δ_+

negative roots

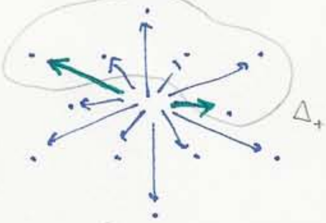
The algebra spanned by the H_i's and by the positive roots is called the Borel subalgebra of $\mathfrak{g}$, and denoted $\mathfrak{b}$.

Other root systems include



(type B_2)

and



(type G_2)

simple roots α_i

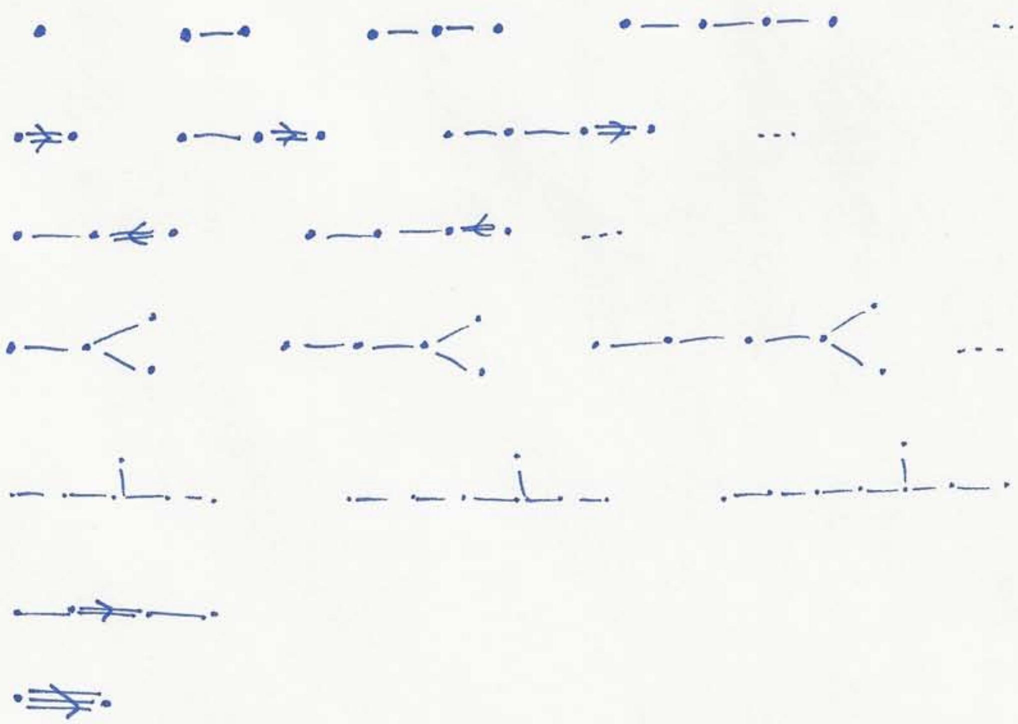
and then there's many more examples, but they live in higher dimensions.

For $sL(2)$, it's

it lives in one dimension.

The way to build a general f.d. simple Lie algebra is as follows:

Start with one of the following "Dynkin diagrams":



A_n

B_n

C_n

D_n

E_6 E_7 E_8

F_4

G_2

and define numbers d_i , a_{ij} as follows ($i,j \in$ vertices)

indicates whether α_i is a long root or a short root — "Cartan matrix" — inner product between simple roots

i j	d_i	a_{ij}	$\langle \alpha_i , \alpha_j \rangle := d_i\, a_{ij}$
$i = j$		2	$2 d_i$
$\bullet \quad \bullet$		0	0
$\bullet\!-\!\!\!-\!\bullet$	$d_i = d_j$	-1	$-d_i$
$\bullet \Rightarrow \bullet$	2	-1	-2
$\bullet \Rrightarrow \bullet$	3	-1	-3
$\bullet \Leftarrow \bullet$	1	-2	-2
$\bullet \Lleftarrow \bullet$	1	-3	-3

The Lie algebra $\mathfrak{g}$ that corresponds to the given Dynkin diagram is then given by the following presentation:

<u>generators</u>: $E_1, E_2, E_3, \ldots E_n,\ F_1, F_2, \ldots F_n,\ H_1, H_2, \ldots H_n$.

$n = \#$ of vertices (= dimension in which the root system lives) = "the <u>rank</u> of $\mathfrak{g}$"

<u>Serre relations</u>:

$$[H_i, H_j] = 0 \qquad [E_i, F_i] = H_i$$
$$[H_i, E_j] = a_{ij} E_j \qquad [E_i, F_j] = 0 \quad (i \neq j)$$
$$[H_i, F_j] = -a_{ij} F_j \qquad \mathrm{ad}(E_i)^{|a_{ij}|+1}(E_j) = 0$$
$$\mathrm{ad}(F_i)^{|a_{ij}|+1}(F_j) = 0$$

Let's decode those relations:

I've drawn everything on a lattice, that's the <u>weight lattice</u> Λ

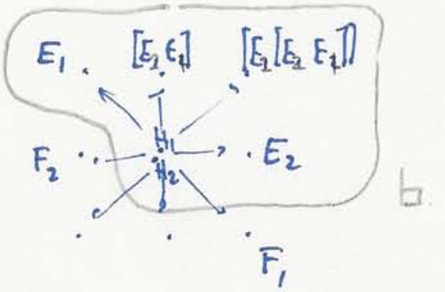
Everything in sight has a given weight (which is an element of Λ).

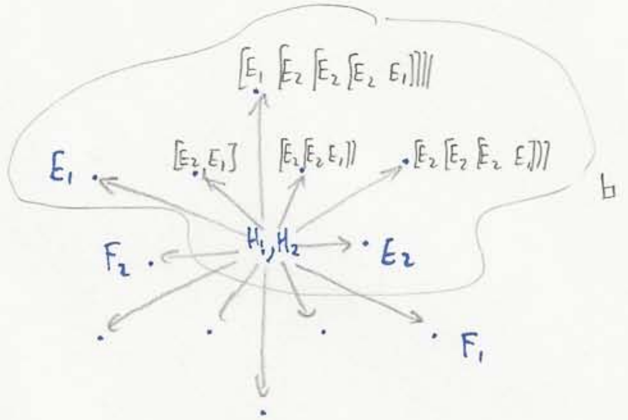
The elements H_i are the things that tell you what your weight is.

(e.g. in the $sl(3)$ picture, if you know the action of H_1 and of H_2, you know where you sit on this lattice)

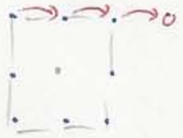
The first three relations are there to tell you the weights of E_i and F_i

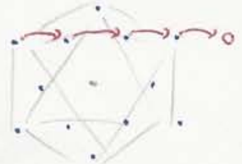
examples:





$\mathrm{ad}(E_i)^{|a_{ij}|+1}(E_j)$: how many times can I apply $[E_i, -]$ before I get 0?





that's what the numbers a_{ij} (Cartan matrix) mean.

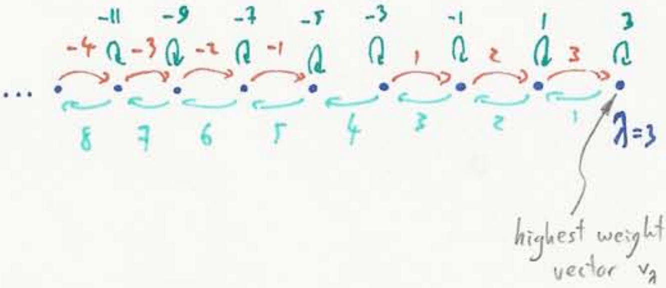
So far, I've told you about Lie algebras and their modules in a qualitative way. Now I want to start with a Lie algebra (coming from a Dynkin diagram) and construct all of its representations.

On our way towards constructing the finite dimensional reps, it turns out to be convenient to first build certain infinite dimensional reps: Verma modules for $\lambda \in \Lambda$ $\lambda = (\lambda_1, \dots, \lambda_n)$ basis of "simple weights"

$$\boxed{M_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda}$$

where $\mathbb{C}_\lambda$ is the 1-dimensional vector space $\mathbb{C}$ with $\mathfrak{b}$-module structure where the H_i act by λ_i and the E_i act by zero

Here is what Verma modules for $sl(2)$ look like



for $\lambda \geq 0$

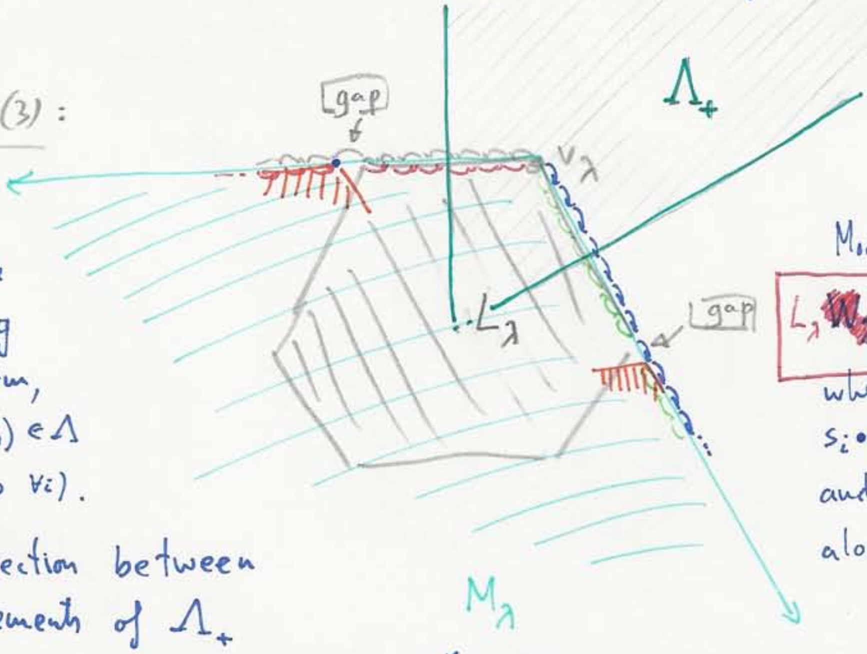
and for $\lambda < 0$, it looks the same except there is no gap.

Let $n_- \subset g$ be the subalgebra generated by the F_i, so that $g = \mathfrak{b} \oplus n_-$.

The Verma module M_λ is "freely generated by n_- acting on v_λ".

The finite dimensional module L_λ is the quotient of M_λ by everything that can't come back to v_λ by means of the raising operator E_i.

Example $sl(3)$:



Thm: Every irreducible representation of g is of that form, with $\lambda = (\lambda_1, \dots \lambda_n) \in \Lambda$ positive (i.e. $\lambda_i \geq 0 \ \forall i$).

I.e. there is a bijection between irreps of g and elements of Λ_+

More precisely:

$$\boxed{L_\lambda = M_\lambda / Span(M_{s_i \bullet \lambda})}$$

where $s_i \bullet \lambda = s_i \lambda - \alpha_i$ and s_i is the reflection along α_i.

We'll be quite interested in characters of representations

Def V g-rep., $V = \bigoplus_{\mu \in \Lambda} V_\mu$ let $\mathfrak{h} := (\Lambda \otimes \mathbb{R})^* = span(H_1, \dots, H_n)$

character of V: $\chi(V) \in Fun(\mathfrak{h}, \mathbb{C})$

$$(\chi(V))(t) := \sum_{\mu \in \Lambda} \dim(V_\mu)\, e^{\langle \mu, t \rangle}$$

If V is infinite dimensional, the sum might not always converge, and so we extend the definition to

$$\chi(V)(t) = \underset{t' \to t}{\text{anal. cont}} \sum_{\mu \in \Lambda} \dim(V_\mu)\, e^{\langle \mu, t' \rangle}$$

For short: $\boxed{\chi(V) = \sum_{\mu \in \Lambda} \dim(V_\mu)\, e^{\mu}}$

The characters of Verma modules are the easiest to compute.

There is a version of the PBW theorem that says that M_λ looks just like $\mathrm{Sym}(\mathfrak{n}_-)$... as far as the dimensions of the graded pieces are concerned. (there is a filtration on M_λ whose associated graded is $\mathrm{Sym}(\mathfrak{n}_-)$)

... with grading shifted by λ:

$$\boxed{\begin{aligned}\chi(M_\lambda) &= e^\lambda \chi(\mathrm{Sym}\,\mathfrak{n}_-) \\ &= e^\lambda \prod_{\alpha\in\Delta_+}(1+e^{-\alpha}+e^{-2\alpha}+\dots) = e^\lambda \prod_{\alpha\in\Delta_+}\frac{1}{1-e^{-\alpha}}\end{aligned}}$$

Extend the notation

Let (L_λ to denote the quotient of M_λ by everything that cannot go back to v_λ (~~dim~~ $\dim(L_\lambda) < \infty$ for $\lambda \in \Lambda_+$, and $\dim(L_\lambda) = \infty$ otherwise))

$$L_\lambda = M_\lambda \Big/ \underset{\{i:\, s_i\bullet\lambda<\lambda\}}{\mathrm{Span}} (M_{s_i\bullet\lambda})$$

W : Weyl group

Letting W be the group generated by the simple reflections s_i, one can show that all the irreducible representations that show up in the decomposition series of M_λ are of the form $L_{\lambda'}$ for $\lambda' = w\bullet\lambda$ and $\lambda' \leq \lambda$

$$\Rightarrow \chi(M_\lambda) = \sum_{\substack{\lambda'=w\bullet\lambda \\ \lambda'<\lambda}} b_{\lambda\lambda'}\, \chi(L_{\lambda'}) \qquad \text{and } b_{\lambda\lambda}=1$$

By inverting the upper triangular matrix $(b_{\lambda\lambda'})$, one gets that

$$\chi(L_\lambda) = \sum_{\substack{\lambda' = w\bullet\lambda \\ \lambda' < \lambda}} a_{\lambda\lambda'}\, \chi(M_{\lambda'}) \qquad \left(\begin{matrix}\text{and} \\ a_{\lambda\lambda} = 1\end{matrix}\right)$$

Now let's take $\lambda \in \Lambda_+$ (so that $L_\lambda = W_\lambda$ is finite dimensional)

then the character of L_λ is Weyl group symmetric.

$$\chi(\cancel{W_\lambda} L_\lambda) = \sum_{\lambda' = w\bullet\lambda} a_{\lambda\lambda'}\, \chi(M_{\lambda'}) = \sum_{w \in W} a_w\, e^{w\bullet\lambda} \prod_{\alpha \in \Delta_+} \frac{1}{1-e^{-\alpha}}$$

$$\overset{?}{=} s_i \left(\sum_w a_w\, e^{w\bullet\lambda} \prod_\alpha \frac{1}{1-e^{-\alpha}} \right)$$

$$= \sum_w a_w\, e^{s_i(w\bullet\lambda)} \prod_{\alpha\in\Delta_+} \frac{1}{1-e^{-s_i\alpha}}$$

$$= \sum_w a_w\, e^{s_i\bullet(w\bullet\lambda)+\alpha_i} \frac{1}{1-e^{\alpha_i}} \prod_{\alpha \in \Delta_+\setminus\{\alpha_i\}} \frac{1}{1-e^{-\alpha}}$$

$$= \sum_w a_w\, e^{(s_i w)\bullet\lambda} \frac{\alpha_i}{1-e^{\alpha_i}} \prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}} \frac{1}{1-e^{-\alpha}}$$

$$= (-1)\cdot \sum_w a_{s_i w}\, e^{w\bullet\lambda} \prod_{\alpha\in\Delta_+} \frac{1}{1-e^{-\alpha}}$$

$\therefore\ a_{s_i w} = -a_w$

$\Rightarrow\ a_w = (-1)^{\text{length}(w)} \equiv (-1)^w$

Weyl character formula:

$$\boxed{\chi(L_\lambda) = \frac{\sum\limits_{w\in W} (-1)^w \, e^{w\bullet\lambda}}{\prod\limits_{\alpha\in\Delta_+} (1-e^{-\alpha})}}$$

By taking $\lambda = 0 \in \Lambda_+$ we also learn that $\prod\limits_{\alpha\in\Delta_+}(1-e^{-\alpha}) = \sum\limits_{w\in W}(-1)^w e^{w\bullet 0}$,
which leads to an alternative formula for $\chi(L_\lambda)$.

The shifted Weyl group action is sometimes inconvenient...

Let $\rho\in\Lambda$ be such that $w\bullet(-\rho) = -\rho$ $\forall w\in W$ [Alternative definition: $\rho = \frac{1}{2}\sum\limits_{\alpha\in\Delta_+}\alpha$]

Then $w\bullet\lambda = w(\lambda+\rho)-\rho$.

$$\boxed{\chi(L_\lambda) = \frac{\sum\limits_{w\in W}(-1)^w e^{w(\lambda+\rho)}}{\prod\limits_{\alpha\in\Delta_+} e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}} = \frac{\sum\limits_{w}(-1)^w e^{w(\lambda+\rho)}}{\sum\limits_{w}(-1)^w e^{w\rho}}}$$

We've seen that irreps of $\mathfrak{g}$ are classified by Λ_+

In particular, there are ∞-ly many of them.

But we know that for the purpose of constructing 3-manifold invariants, we need tensor categories with finitely many objects.

This will be acheived in two steps

① ~~Deforming~~ $\mathfrak{g}$ (actually that's not possible)

Deforming $U\mathfrak{g} \rightsquigarrow U_q\mathfrak{g}$

② Letting q be a root of unity.

The algebra $U\mathfrak{g}$ has same presentation as $\mathfrak{g}$
(except that now it's an associative algebra, and not a Lie algebra)

Thinking of H_j as functions on Λ, the ~~first three~~ relations

$\cancel{[H_i, H_j] = 0}$, $[H_i, E_j] = a_{ij} E_j$; $[H_i, F_j] = -a_{ij} F_j$

can be rewritten

$$\boxed{f \cdot E_i = E_i \cdot \tau_{\alpha_i}(f)}$$

where $\tau_{\alpha_i}(f)(x) = f(x+\alpha_i)$

In $U_q\mathfrak{g}$, we'll be working with the functions $K_i := q^{d_i H_i}$ instead of H_i

Here, q is an indeterminate, and our base field is $\mathbb{C}(q)$.

The relation between K_i and E_j & F_j then become

$$K_i E_j = q^{d_i a_{ij}} E_j K_i$$
$$K_i F_j = q^{-d_i a_{ij}} F_j K_i$$

or equivalently

$$K_i E_j K_i^{-1} = q^{\langle \alpha_i, \alpha_j \rangle} E_j$$
$$K_i F_j K_i^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle} F_j$$

Now we need to see what to do with the other relations

- $[E_i, F_j] = \delta_{ij} H_i$
- $\displaystyle\sum_{r=0}^{|a_{ij}|+1} (-1)^r \binom{|a_{ij}|+1}{r} E_i^{|a_{ij}|+1-r} E_j E_i^r = 0$
- (idem for F)

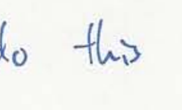
---

q-deformations:

$$(X+Y)^n = \sum_{r=0}^{n} \binom{n}{r} X^{n-r} Y^r$$

Let's redo this with $XY = qYX$:

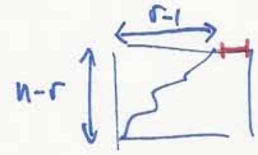
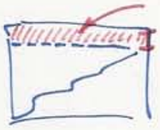
$$(X+Y)^2 = X^2 + XY + YX + Y^2 = X^2 + (1+q)XY + Y^2$$
$$(X+Y)^3 = X^3 + (1+q+q^2)X^2Y + (1+q+q^2)XY^2 + Y^3$$

If I draw each monomial in $(X+Y)^n$ contributing to $X^{n-r}Y^r$ by a staircase  then the exponent of q that it gets is the shaded area.

One of the possible definitions of quantum binomial coefficients is by the formula

$$(X+Y)^n = \sum \begin{bmatrix} n \\ r \end{bmatrix}_q X^{n-r} Y^r \qquad (XY = qYX)$$

and they satisfy the recurrence $\begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}$.

combinatorially: a path  either starts   or starts 

q-Pascal triangle:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 1+q & & 1 & & \\
 & 1 & & 1+q+q^2 & & 1+q+q^2 & & 1 & \\
1 & & 1+q+q^2+q^3 & & 1+q+2q^2+q^3+q^4 & & 1+q+q^2+q^3 & & 1
\end{array}$$

We'll prefer a more symmetric version of the q-binomial coefficients:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & q^{-1}+q & & 1 & \\
1 & & q^{-2}+1+q^2 & & q^{-2}+1+q^2 & & 1 \\
\vdots & & & & & & \vdots
\end{array}$$

recurrence:

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = q^{-(n-r)} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}$$

Let's define $$[n]_q := q^{-n+1} + q^{-n+3} + \dots + q^{n-1} = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$[n]_q! := [1]_q [2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{[n]_q!}{[r]_q! \, [n-r]_q!}$$

then $\begin{bmatrix} n \\ r \end{bmatrix}_q$ satisfies the above recursion relation :

$$\frac{[n]!}{[r]![n-r]!} \overset{?}{=} q^{-(n-r)} \frac{[n-1]!}{[r-1]!\,[n-r]!} + q^{r} \frac{[n-1]!}{[r]!\,[n-r-1]!}$$

$$\times \frac{[r]!\,[n-r]!}{[n-1]!}$$

$$\Leftrightarrow \quad [n] \overset{?}{=} q^{-(n-r)} [r] + q^{r} [n-r] \qquad \checkmark$$

When q^2 is an ℓ-th root of unity then the above quantities have extra symmetries:

$$[n] = [\ell - n]$$

$\ell = 7$:

$[3] = [4]$

by applying the above to all the terms in the numerator

and $$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n][n-1]\dots[n-r+1]}{[r]!} \overset{\ell}{=} \begin{bmatrix} \ell + r - n - 1 \\ r \end{bmatrix}$$

<u>Example</u> The ℓ first rows of Pascal's triangle when q^2 is of order ℓ:

In these lectures, we'll always assume the order of q is <u>even</u>

ℓ:

$\ell = 1$:
```
1
```

$\ell = 2$:
```
 1
1 1
```

$\ell = 3$:
```
  1
 1 1
1 1 1
```

$\ell = 4$:
```
   1
  1 1
 1 √2 1
1 1 1 1
```

$\ell = 5$:
```
    1
   1 1
  1 φ 1
 1 φ φ 1
1 1 1 1 1
```

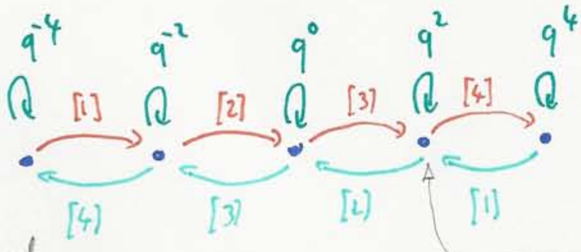
$\ell = 6$:
```
     1
    1 1
   1 √3 1
  1 2 2 1
 1 √3 2 √3 1
1 1 1 1 1 1
```

Back to quantum groups.

Quantum Serre relations:

- $K_i K_j = K_j K_i$
- $K_i E_j K_i^{-1} = q^{\langle \alpha_i \alpha_j \rangle} E_j$
- $K_i F_j K_i^{-1} = q^{-\langle \alpha_i \alpha_j \rangle} F_j$
- $E_i F_j - F_j E_i = \delta_{ij} [H_i]_{q^{d_i}} = \dfrac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}$
- $\displaystyle\sum_{r=0}^{|a_{ij}|+1} (-1)^r \begin{bmatrix} |a_{ij}|+1 \\ r \end{bmatrix}_{q^{d_i}} E_i^{|a_{ij}|+1-r} E_j E_i^r = 0$
- (idem for F)

Modules for $U_q(sl(2))$:



K E F

check: $\underbrace{EF}_{=[3][2]} - \underbrace{FE}_{=[1][4]} \overset{?}{=} \underbrace{\frac{K-K^{-1}}{q-q^{-1}} = [H]}_{=[2]}$

Just like in the classical case, we also have Verma modules $M_\lambda = U_q(\mathfrak{g}) \otimes_{U_q\mathfrak{b}} \mathbb{C}_\lambda$ (story looks the same)

unfortunately, it is no longer true that all f.d. modules are as above.

One also has $K \to -K$ (and then that's all).
$E \to -E$
$F \to F$

But we don't like those other modules, so we're going to pretend that they don't exist, and define our category of rep's as containing only the first kind.

So far, I have told you what $U_q(\mathfrak{g})$ is as an algebra. Now we want to make it into a <u>Hopf algebra</u>

We need

$$\Delta : A \to A \otimes_{\mathbb{C}(q)} A$$
$$\varepsilon : A \to \mathbb{C}(q)$$

algebra homomorphism

$$S : A \to A$$

algebra anti-homomorphism

($\therefore$ it's enough to give these on generators)

Recall:
- $K_i K_j = K_j K_i$
- $K_i E_j K_i^{-1} = q^{\langle \alpha_i, \alpha_j \rangle} E_j$
- $K_i F_j K_i^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle} F_j$
- $E_i F_j - F_j E_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}$
- $\sum_{r=0}^{|a_{ij}|+1} (-1)^r \left[\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right] E_i^{\cdots} E_j E_i^{\cdots}$
- (same for F_i)

$\Delta(K_i) = K_i \otimes K_i$	$\varepsilon(K_i) = 1$	$S(K_i) = K_i^{-1}$
$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i$	$\varepsilon(E_i) = 0$	$S(E_i) = -E_i K_i^{-1}$
$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$	$\varepsilon(F_i) = 0$	$S(F_i) = -K_i F_i$

Now there's a lot to check in order to see that this indeed a Hopf algebra...

- check that these prescriptions <u>define</u> algebra (anti)-homomorphisms i.e. respect the relations
- check $\Delta \circ S = (S \otimes S) \circ \Delta^{op}$ $\leftarrow$ because both LHS and RHS are algebra anti-homomorphisms, it's enough to check this on generators
- check $m \circ (S \otimes 1) \circ \Delta = m \circ (1 \otimes S) \circ \Delta = \eta \circ \varepsilon$ $\leftarrow$ Once again, it's enough to check this on generators, (but now the reason is more involved)

I'll check one instance of each one of these:

(1) $\Delta(K_i E_j K_i^{-1}) = (K_i \otimes K_i)(E_j \otimes K_j + 1 \otimes E_j)(K_i^{-1} \otimes K_i^{-1})$

$$= K_i E_j K_i^{-1} \otimes K_j + 1 \otimes K_i E_j K_i^{-1}$$

use the rels we have

$$= q^{\langle \alpha_i \alpha_j \rangle}(E_j \otimes K_j + 1 \otimes E_j) = \Delta\left(q^{\langle \alpha_i \alpha_j \rangle} E_j\right) \quad ✓$$

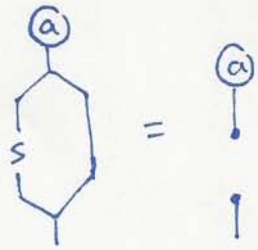
(2) $\Delta \circ S(E_i) = \Delta(-E_i K_i^{-1}) = -(E_i \otimes K_i + 1 \otimes E_i)(K_i^{-1} \otimes K_i^{-1})$

$$= - E_i K_i^{-1} \otimes 1 - K_i^{-1} \otimes E_i K_i^{-1}$$

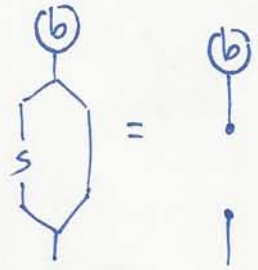
$$= (S \otimes S)(1 \otimes E_i + K_i \otimes E_i) = (S \otimes S) \circ \Delta^{op}(E_i) \quad ✓$$

(3) I'll show why it's enough to check it on generators:

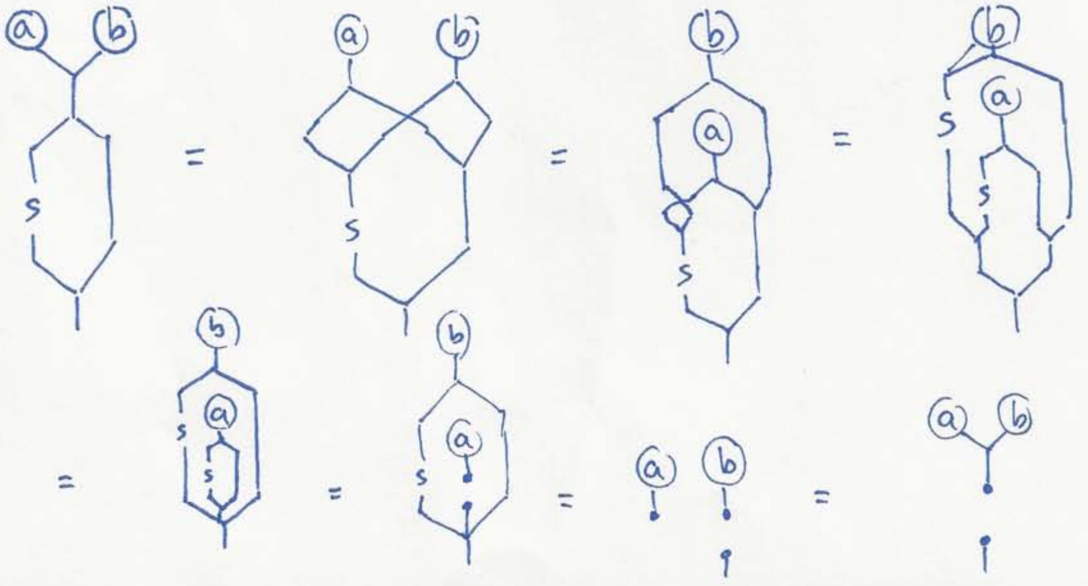
Suppose we know



and



then

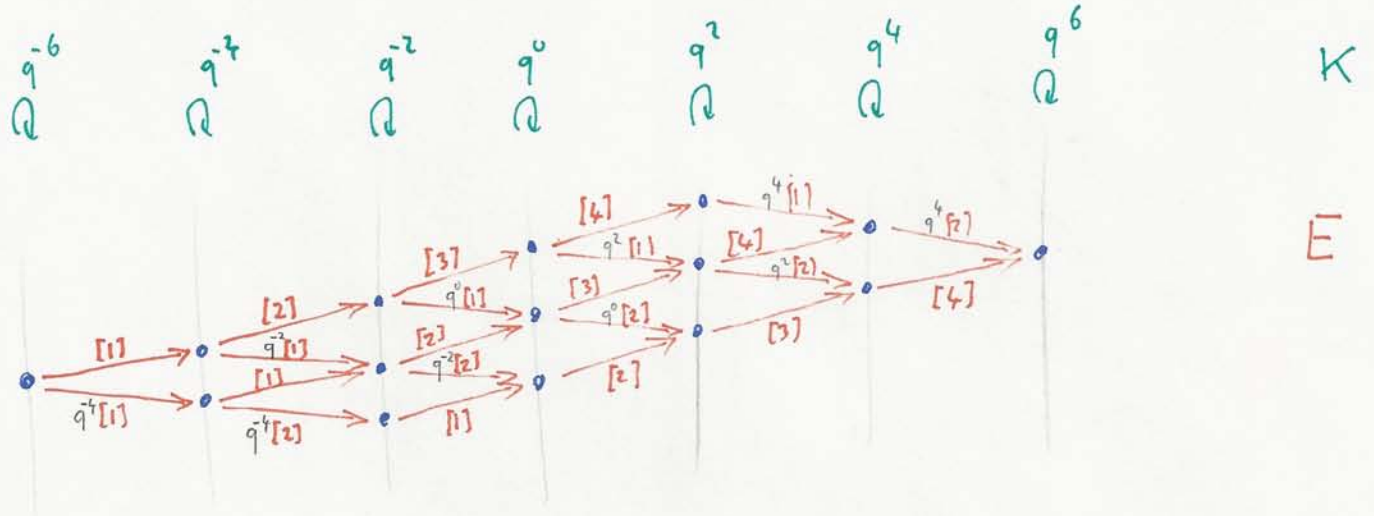


✓

Now that we have a comultiplication, it makes sense to take the $\otimes$ of representations.

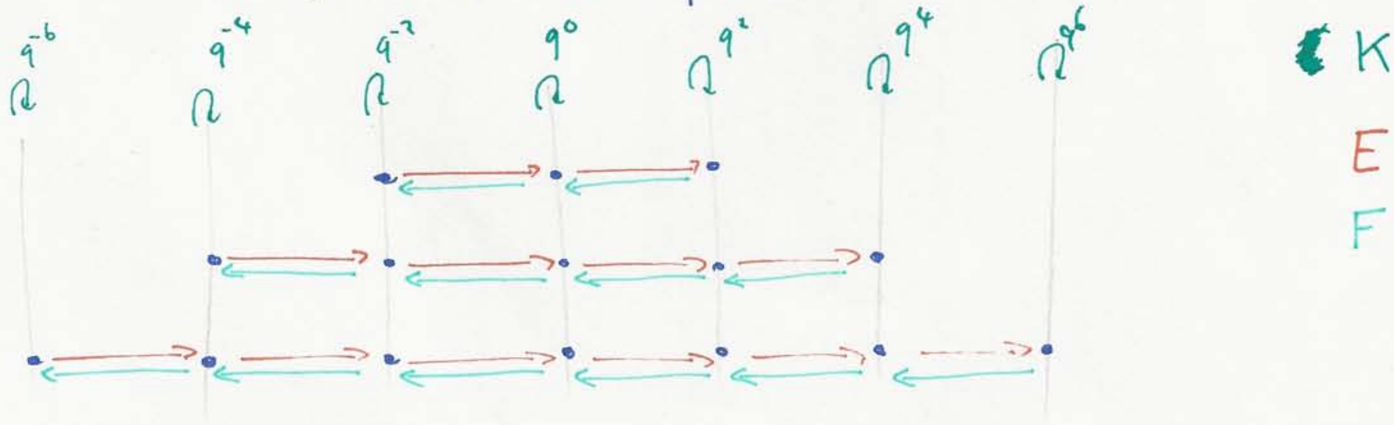
Let's do that in a simple case ($U_q sl(2)$) to see what we get:

(3-dimensional rep) $\otimes$ (5 dimensional rep):



if the coproduct was $\Delta(E) = E\otimes 1 + 1\otimes E$ as in $U(\mathfrak{g})$ then

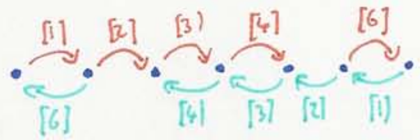
when q is not a root of unity, nothing unexpected happens, and we get a $\oplus$ decomposition



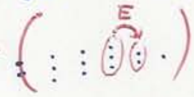
indeed, that's the only way of getting the weight spaces of dim 1233321.

But when q is a root of unity, something strange happens. Let's take the above example and assume q^2 has order 5. (level 3)

Then the seven dimensional summand looks like this:

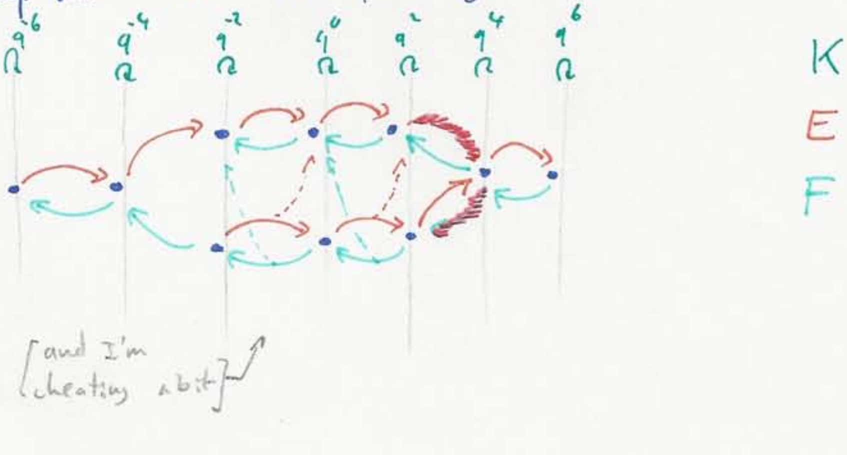


In particular, if we had such a $\oplus$ decomposition for (3-dim rep) $\otimes$ (5-dim rep) then the result would not be self-dual [the dual of  is ]

but that's impossible because both (3-dim rep) and (5-dim rep) are self-dual, and so their tensor product must also be. Another argument: ker  is one dimensional. If it was $3 \oplus 5 \oplus 7$ then that kernel would be 2 dim.

The structure of this $\otimes$ turn out to be a little bit more complicated:

- the 5-dim summand stays the same
- the 10-dim complement has the following structure:



[and I'm cheating a bit]

That's an example of a __tilting module__.

Now let me go back to the general case and give some definitions:

Verma modules: $M_\lambda = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b})} \mathbb{C}_\lambda$

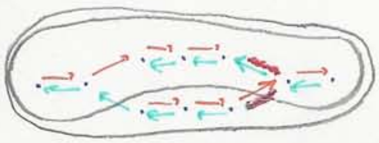
Weyl modules: $W_\lambda = M_\lambda / \mathrm{span}(M_{s_i \cdot \lambda})$ $\quad \lambda \in \Lambda_+$

[I don't use the notation L_λ because that notation implicitely mean that the module is irreducible, and we've seen that when q is a root of unity, that no longer needs to be the case]

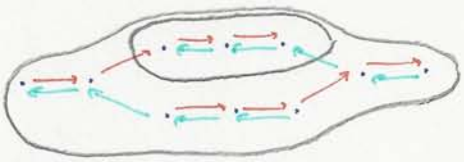
dual Weyl modules: W_λ^*

Definition: A tilting module is a module T that admits a filtration whose associated graded pieces are Weyl modules, and also admits a filtration whose associated graded pieces are dual Weyl modules

example:



Weyl filtration



dual Weyl filtration

When we were studying Verma modules for the classical Lie algebra case, it turned out to be very useful to know which other Verma modules one could find inside a given Verma module M_λ,

and the answer turned out to be $M_{w\cdot\lambda}$ (shifted Weyl group action)

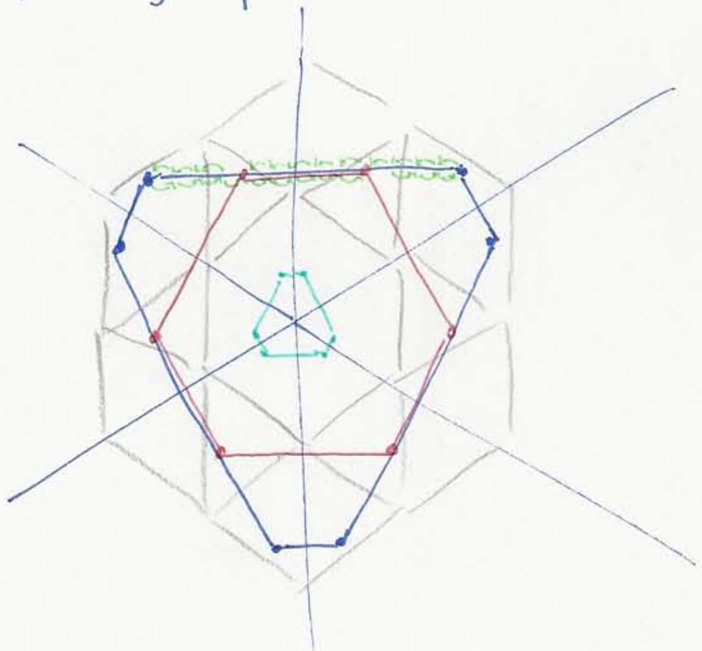
"linkage principle"

The corresponding question that is relevant for quantum groups is the following:

In a given Weyl module W_λ, which other Weyl modules can one find inside it?

(if q is not a root of unity, then W_λ is irreducible, and the ~~answ~~ is "nothing")

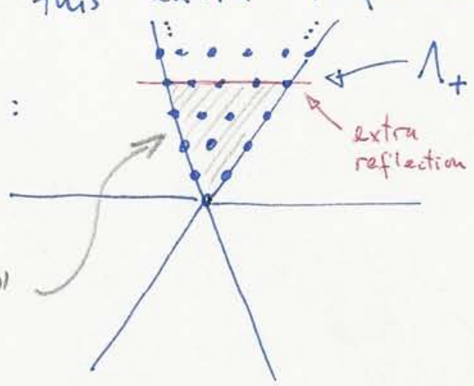
Let me tell you what the answer is in the case of $sL(3)$ by drawing a picture:



[I'm lying a bit: one needs a shifted action of the affine Weyl group]

Answer $W_{\lambda'} \subset W_\lambda$ if λ' and λ are related by ~~an~~ action of the Affine Weyl group := group generated by usual Weyl group and an extra reflection

The location of this extra reflection depends on the order of q^2:



"Weyl alcove"

for each $\lambda \in \Lambda_+$ we have a corresponding Weyl module W_λ.

If λ is in the alcove then W_λ is irreducible, and if λ is outside the alcove, then W_λ is not irreducible.

(and how big the alcove is depends on the order of q^2)

<u>The plan</u>:

$U_q(\mathfrak{g})$ (q not a root of 1) — same rep theory as $\mathfrak{g}$

$\rightsquigarrow$

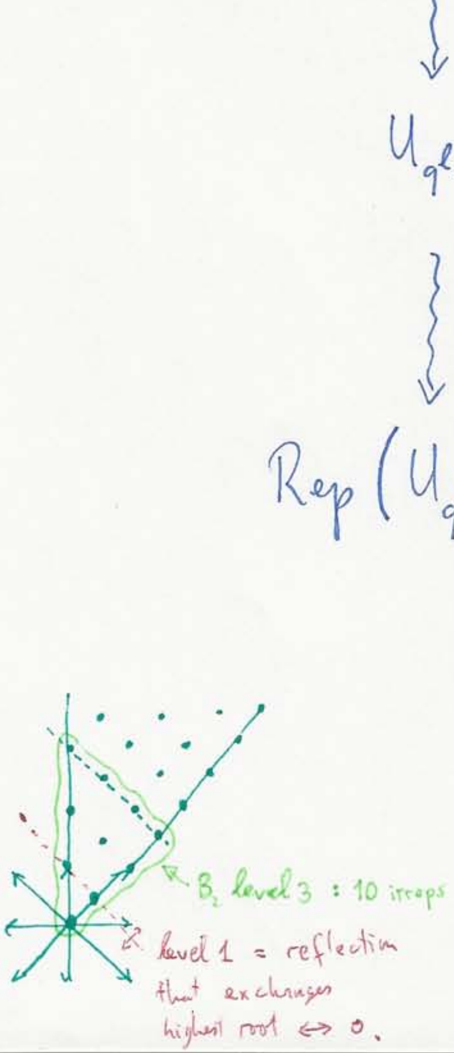
$U_{q^\ell = 1}(\mathfrak{g})$ — representation theory is not semi-simple

$\rightsquigarrow$

$\mathrm{Rep}\left(U_{q^\ell=1}(\mathfrak{g})\right)/\sim$ — mod out by all the non-semisimple stuff

actually that's a lie: mod out everything with quantum dimension $= 0$ [includes the non-semisimple stuff and a little bit more]

We're left with a semisimpl $\otimes$ category with finitely many simple objects, indexed by the weights $\lambda \in \Lambda$ that are in the Weyl alcove.



We have constructed $U_q(\mathfrak{g})$ as a Hopf algebra, and so $\mathrm{Rep}(U_q(\mathfrak{g}))$ is a tensor category. I told you a little bit what will happen to $\mathrm{Rep}(U_q(\mathfrak{g}))$ once we set $q^{2\ell} = 1$, but we need to first understand better the structure that's there for generic q before we can deal with the root of unity case.

<u>R-matrix</u> (= braided structure on $\mathrm{Rep}(U_q(\mathfrak{g}))$)

there's a trick: there's a Hopf ~~[illegible]~~ pairing between $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)^{\overline{op}}$ (opposite coproduct)

<u>Definition</u> A, B Hopf algebras

$\langle \cdot, \cdot \rangle : A \otimes B \to \mathbb{C}$ is a Hopf pairing if

$\langle a, b_1 b_2 \rangle = \langle \Delta(a), b_1 \otimes b_2 \rangle$ and $\langle a, 1 \rangle = \varepsilon(a)$, $\langle 1, b \rangle = \varepsilon(b)$.

$\langle a_1 a_2, b \rangle = \langle a_1 \otimes a_2, \Delta(b) \rangle$ (the RHS uses the pairing $(A \otimes A) \otimes (B \otimes B) \to \mathbb{C}$ given by $\langle a \otimes a', b \otimes b' \rangle := \langle a, b \rangle \langle a', b' \rangle$)

Let $e_p \in U_q(\mathfrak{b}_+)$ be a basis and let $e^p \in U_q(\mathfrak{b}_-)$ be the dual basis w.r.t. the above pairing

If the multiplication and coproduct in $U_q(b_+)$ are given by

$$e_s e_t = \sum_p \mu^p_{st} e_p$$

$$\Delta(e_p) = \sum_{s,t} \nu^{st}_p e_s \otimes e_t$$

then the multiplication and coproduct in $U_q(b_-)$ are given by

$$e^s e^t = \sum_p \nu^{st}_p e^p$$

$$\Delta(e^p) = \sum_{s,t} \mu^p_{ts} e^s \otimes e^t$$

Claim: $R := \sum_p e_p \otimes e^p$ is an R-matrix (formally)

proof: $(\Delta \otimes id) R = \sum_p \Delta(e_p) \otimes e^p$

$$= \sum_{p,s,t} \nu^{st}_p e_s \otimes e_t \otimes e^p$$

$$= \sum_{s,t} e_s \otimes e_t \otimes e^s e^t = R_{13} \cdot R_{23} \quad \checkmark$$

Oops! I seem to have forgotten $\Delta^{op}(a) = R \Delta(a) R^{-1}$

$$(id \otimes \Delta) R = \sum_p e_p \otimes \Delta(e^p)$$

$$= \sum_{p,s,t} \mu^p_{st} e_p \otimes e^t \otimes e^s$$

$$= \sum_{s,t} e_s e_t \otimes e^t \otimes e^s = R_{13} \cdot R_{12} \quad \checkmark$$

This all looks very easy, but I should warn you that there are issues with the interpretation of $\sum_p e_p \otimes e^p$ because the algebras $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ are not finite dimensional...

We're getting ahead of ourselves... let's first write the <u>Hopf pairing</u>:

$$\langle E_i , F_i \rangle = \frac{1}{[d_i]_q}$$

$$\langle K_i , K_j \rangle = q^{\langle \alpha_i, \alpha_j \rangle}$$

and all the other pairings between generators are zero.

Reason is that the pairing is compatible with the grading by Λ and only allows $\neq 0$ value when the weights add up to zero

I claim that it's enough to give the pairing on generators to determine the whole thing

<u>example</u>: $SL(2)$: $\langle E, F \rangle = 1$, $\langle E, K \rangle = 0$, $\langle K, F \rangle = 0$, $\langle K, K \rangle = q^2$

recall: $\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes K + 1 \otimes E$, $\Delta^{op}(F) = F \otimes K^{-1} + 1 \otimes F$

$$\begin{aligned}
\langle E^2, F^2 \rangle &= \langle \Delta(E^2), F \otimes F \rangle \\
&= \langle \Delta(E) \cdot \Delta(E), F \otimes F \rangle \\
&= \langle (E \otimes K + 1 \otimes E)(E \otimes K + 1 \otimes E), F \otimes F \rangle \\
&\overset{\boxed{\text{bi}}\text{degree reasons}}{=} \langle E \otimes KE + E \otimes EK, F \otimes F \rangle \\
&= \langle KE, F \rangle + \langle EK, F \rangle \\
&= \langle K \otimes E, 1 \otimes F \rangle + \langle E \otimes K, F \otimes K^{-1} \rangle \\
&= \langle K, 1 \rangle + \langle K, K^{-1} \rangle = 1 + q^{-2}
\end{aligned}$$

exercise: $\langle K^n, K^m \rangle = q^{2nm}$

Hopefully you get an idea of how to determine $\langle \cdot , \cdot \rangle$ given the values on the generators.

Less clear is why this is well defined. ← It's not just unclear: it's hard. No general methods that allow you to deal with such a situation.

(let's take that for granted and push on)

Way to go: construct an explicit basis of $U_q(b_+)$ and $U_q(b_-)$ and check everything directly

---

Back to $R = \sum_p e_p \otimes e^p$

we'll soon see that it's totally not obvious how to interpret that expression, so let's not accumulate difficulties and do $\mathfrak{sl}(2)$ first:

basis of $U_q(\mathfrak{b}_+)$: $\{K^n E^s\}_{\substack{n \in \mathbb{Z} \\ s \in \mathbb{N}}}$ (*) & of $U_q(\mathfrak{b}_-)$: $\{K^m F^t\}_{\substack{m \in \mathbb{Z} \\ t \in \mathbb{N}}}$

computation:

$$\langle K^n E^t , K^m F^s \rangle = q^{2nm} \, \delta_{ts} \, q^{-\binom{t}{2}} [t]_q!$$

To compute the dual basis to (*), we need to invert a matrix [the E's and the F's don't cause any problem because, as far as they are concerned, the matrix is diagonal]

But how do you invert the $\mathbb{Z} \times \mathbb{Z}$ matrix whose (m,n) entry is q^{2nm} ???

<u>Idea</u>: let's go back to H and see if things look better (recall: $K = q^H$)

Recall $\langle K^n, K^m \rangle = q^{2nm}$

$$\langle K^n, K^m\rangle = \langle \Delta^m(K^n), \underbrace{K\otimes\ldots\otimes K}_{m}\rangle$$
$$= \langle \underbrace{K^n\otimes\ldots\otimes K^n}_{m}, \underbrace{K\otimes\ldots\otimes K}_{m}\rangle$$
$$= \langle K^n, K\rangle^m = \ldots = \langle K, K\rangle^{nm} = q^{2nm}$$

<u>shot in the dark:</u> there is a meaningful way of extending $\langle \cdot,\cdot \rangle$ to H:

$$a := \langle H, H\rangle \qquad a = ?$$

$$\langle H^n, H^m\rangle = \langle \Delta^m(H^n), \underbrace{H\otimes\ldots\otimes H}_{m}\rangle$$
$$= \Big\langle \Big(\sum_{i=1}^{m} 1\otimes 1\otimes\ldots\otimes \underset{\overset{\uparrow}{i}}{H} \otimes 1\otimes\ldots\otimes 1\Big)^n, \underbrace{H\otimes\ldots\otimes H}_{m}\Big\rangle$$

(WLOG $n\le m$)
$$\overset{\Downarrow}{=} \begin{cases} 0 & n<m \quad \text{because } \langle 1, H\rangle = \varepsilon(H) = 0 \\ n!\, a^n & n = m \end{cases}$$

$$\langle K^i, K^j\rangle = \Big\langle \sum_{n=0}^{\infty} \frac{(i\log(q))^n}{n!} H^n, \sum_{j=0}^{\infty}\frac{(j\log(q))^m}{m!} H^m \Big\rangle$$
$$= \sum_{n=0}^{\infty} \frac{(ij\log(q)^2)}{n!}\, a = \exp(ij\log(q)^2 a)$$
$$= q^{ij\log(q)a} \overset{!}{=} q^{2ij} \qquad \therefore\ a = \frac{2}{\log(q)}$$

$$\therefore\ \langle H^n, H^m\rangle = \delta_{nm}\, n! \left(\frac{2}{\log(q)}\right)^n$$

More generally, $\langle H^n E^t, H^m F^s\rangle = \delta_{nm}\,\delta_{ts}\, n! \left(\frac{2}{\log(q)}\right)^n q^{-\binom{t}{2}} [t]_q!$

Finding a dual basis of $\{H^n E^t\}$ is now possible since the pairing matrix is <u>diagonal</u>: it's given by

$$\left\{ \frac{1}{n!}\left(\frac{1}{2}\log(q)\right)^n q^{\binom{t}{2}} \frac{1}{[t]_q!} H^n F^t \right\}$$

$$\therefore \quad R = \sum_{n,t} \frac{1}{n!}\left(\frac{1}{2}\log(q)\right)^n q^{\binom{t}{2}} \frac{1}{[t]_q!} H^n E^t \otimes H^n F^t$$

$$= \sum_{n,t} \frac{1}{n!}\left(\frac{1}{2}\log(q)\, H\otimes H\right)^n q^{\binom{t}{2}} \frac{1}{[t]_q!} E^t \otimes F^t$$

$$= \boxed{q^{\frac{1}{2} H\otimes H} \sum_{t\geq 0} q^{\binom{t}{2}} \frac{1}{[t]_q!} E^t \otimes F^t}$$

It's not quite an element of $U_q(sl(2)) \otimes U_q(sl(2))$, but given any two modules V and W, the action of R on $V\otimes W$ is well-defined.

Problem: This is all well defined when the base field is $\mathbb{C}(q)$, but when $q^{2\ell}=1$, then $[\ell]_q = 0$, and so we're dividing by zero in the definition of R !

Solution: Change the algebra $U_q(\mathfrak{g})$ by introducing divided powers of E_i and F_i

$$E_i^{(r)} = \frac{E_i^r}{[r]_{q^{d_i}}!}, \qquad F_i^{(r)} = \frac{F_i^r}{[r]_{q^{d_i}}!} \qquad U_q^{res}(\mathfrak{g})$$

actually introduce them as new generators subject to the relations $E_i^{(r)} E_i^{(s)} = \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^{d_i}} E_i^{(r+s)}$.

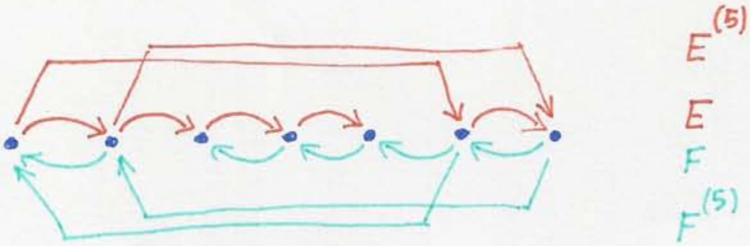
The new R-matrix now looks (back to $sl(2)$) :

$$\boxed{R = q^{\frac{1}{2} H \otimes H} \sum_{t \geq 0} q^{\binom{t}{2}} [t]_q! \; E^{(t)} \otimes F^{(t)}}$$

no more divisions by zero ✓

Now that we changed our algebra $U_q(\mathfrak{g}) \rightsquigarrow U_q^{res}(\mathfrak{g})$, we should go back and check whether our modules changed or not

example ($sl(2)$) q^2 of order 5 :



the Weyl modules are a little bit more connected, but still they have this feature of having a non-trivial submodule in the middle.

So far, we've constructed our Hopf algebra $U_q(\mathfrak{g})$ along with an R-matrix $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.

Actually, we've seen that the formula for R involves dividing by $[t]_q!$ for all $t \in \mathbb{N}$, and when setting q to be a root of unity, this becomes division by zero, so we modified $U_q(\mathfrak{g}) \rightsquigarrow U_q^{res}(\mathfrak{g})$ and everything works fine

$$\boxed{\Longrightarrow \quad \mathrm{Rep}\left(U^{res}_{q^\ell=1}(\mathfrak{g})\right) \text{ is a braided category}}$$

Recall goal: construct $Z : \mathrm{Bord}_1^3 \to \mathrm{LinCat}$

$S^1 \mapsto \mathrm{Rep}\left(U^{res}_{q^\ell=1}(\mathfrak{g})\right)/\sim$

to make it semi-simple & with fin many objects.

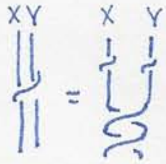
But there's another structure that $Z(S^1)$ has in any (1-3)-TFT : it's a <u>ribbon category</u>. I want to talk about the structure on the Hopf algebra side that will make $\mathrm{Rep}(A)$ into a ribbon category.

<u>Definition</u> Let $\mathcal{C}$ be a braided category with duals. To make $\mathcal{C}$ into a ribbon category, we need an extra piece of data, called the <u>twist</u> $\theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \mathrm{Id}_{\mathcal{C}}$ subject to some axioms.

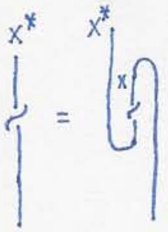
For every object $X \in \mathcal{C}$, we have $\theta_X : X \xrightarrow{\simeq} X$

θ_X is denoted graphically by .

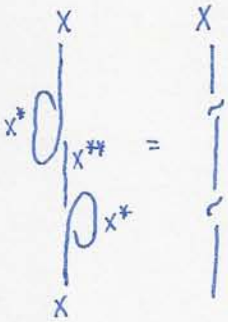
<u>Axioms</u> :



$$\theta_{X\otimes Y} = \beta_{X,Y} \circ \beta_{Y,X} \circ (\theta_X \otimes \theta_Y)$$

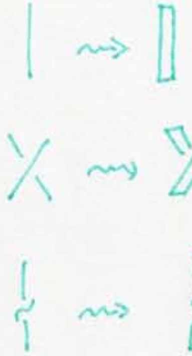


$$\theta_{X^*} = (coev_X \otimes 1)(1 \otimes \theta_X \otimes 1)(1 \otimes ev_X)$$



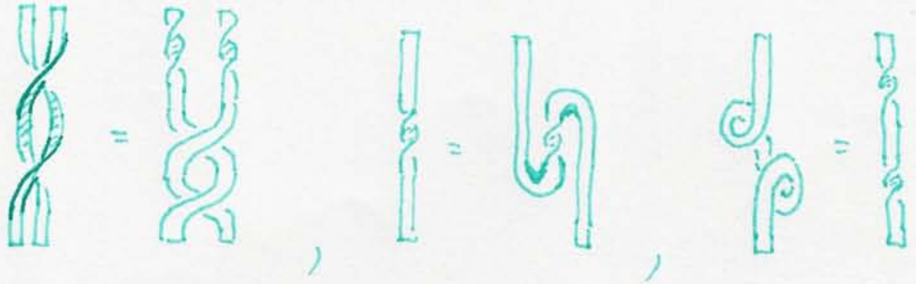
$$(1 \otimes coev_{X^*})(\beta_{X,X^{**}} \otimes 1)(1 \otimes ev_X)(coev_X \otimes 1)(1 \otimes \beta_{X,X^*})(ev_{X^*} \otimes 1) = \theta_X^2$$

Better graphical notation



ribbon

axioms:



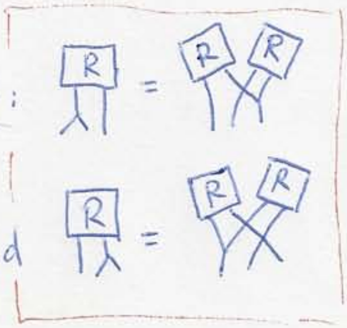
More generally, a ribbon category is made so that any ribbon braid (ribbon tangle) can be interpreted in $\mathcal{C}$ and any isotopy corresponds to an equation between morphisms of $\mathcal{C}$.

<u>Theorem</u> Let A be a braided Hopf algebra equipped with an R-matrix $R \in A \otimes A$

$$\begin{cases} \Delta^{op}(a) = R\,\Delta(a)\,R^{-1} \\ (\Delta \otimes 1)R = R_{13}R_{23} \\ (1 \otimes \Delta)R = R_{13}R_{12} \end{cases}$$

(I completely forgot to prove that relation when I was talking about it yesterday)

graphically:

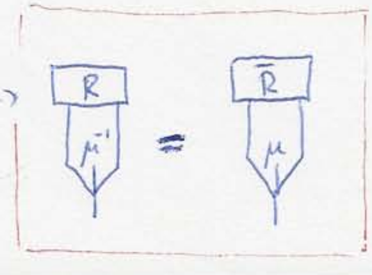


and

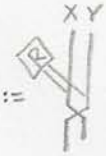
and with a charmed element $\mu \in A$

$$\begin{cases} S^2(a) = \mu\, a\, \mu^{-1} \\ \nu := m(1 \otimes \mu^{-1})\, R = m(1 \otimes \mu)\, \bar{R} \\ S(\mu) = \mu^{-1} \\ \Delta(\mu) = \mu \otimes \mu \end{cases}$$

ribbon element

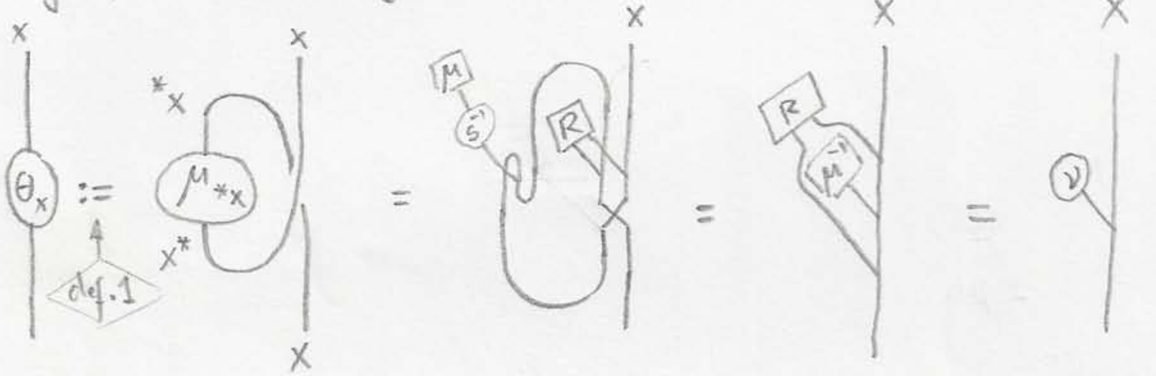


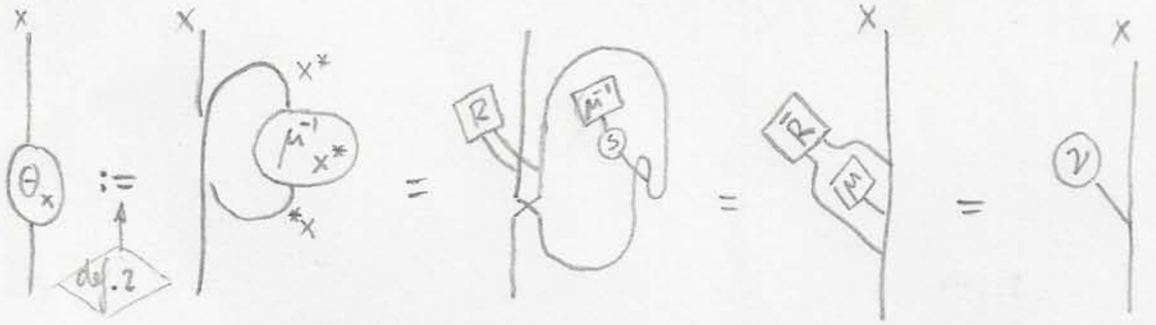
then Rep(A) is a ribbon category.

Before diving into the proof, let me comment on the meaning of some of these equations: $\Delta^{op}(a) = R\,\Delta(a)\,R^{-1}$: $\beta_{XY} :=$  is a module homomorphism $X \otimes Y \to Y \otimes X$

$S^2(a) = \mu\, a\, \mu^{-1}$: $\mu_X :=$  is a module homomorphism $X \to X^{**}$

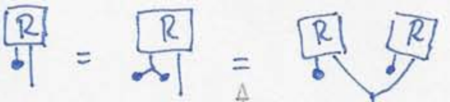
$*$ says that the two possible definitions of the twist agree:

(1)


(2)


Lemma 1: $(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1$

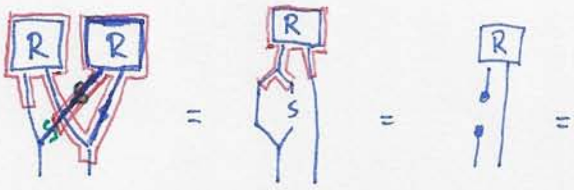
proof: $(\varepsilon \otimes \mathrm{id})(R)$ is invertible and equal to its square

 ☒ (other equation is similar)

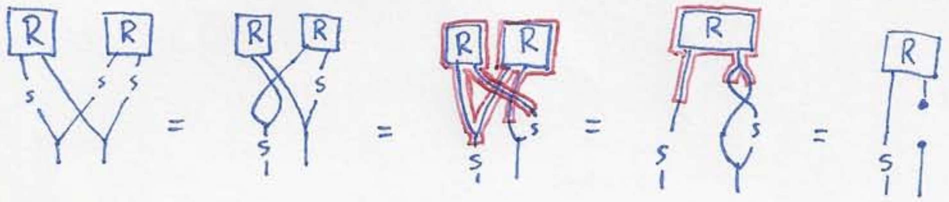
uses


Lemma 2: $(S\otimes S)(R) = R$

proof: $(S\otimes 1)(R)$ is a right inverse of R and a left inverse of $(S\otimes S)(R)$



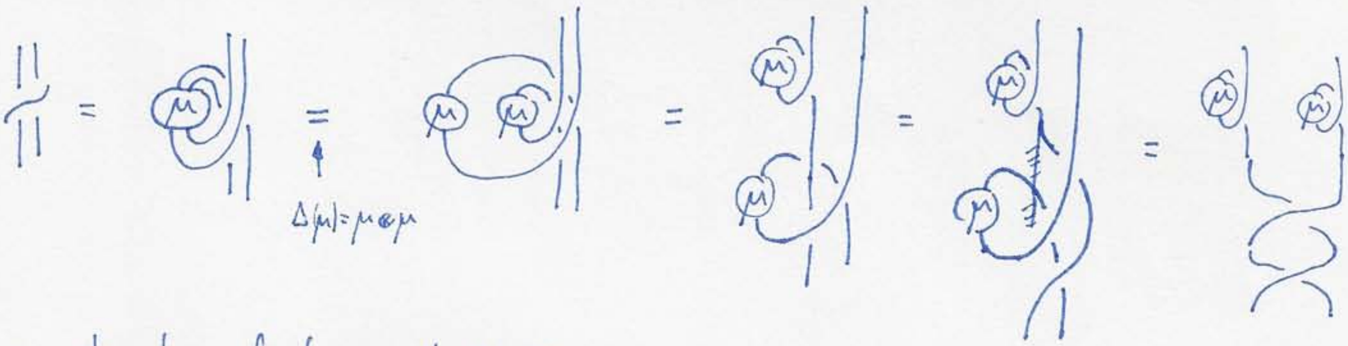
$= 1\otimes 1$



$= S(1)\otimes 1 = 1\otimes 1$

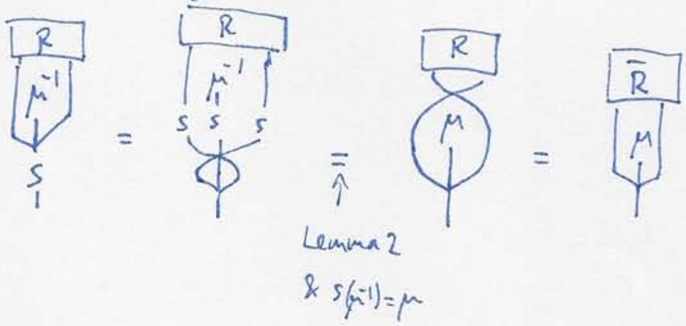
☒

Proof of the theorem:

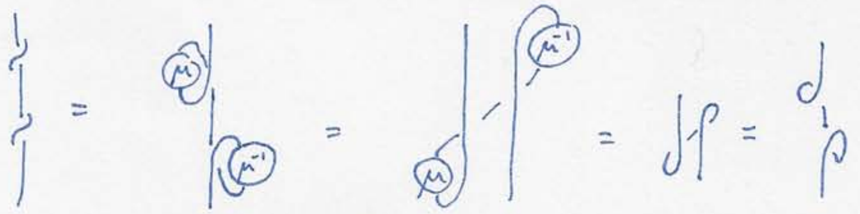
① 

✓

② amounts to checking that $S(\nu) = \nu$

$S(\nu) =$ 

$= \nu$

③ 

☒

We know that

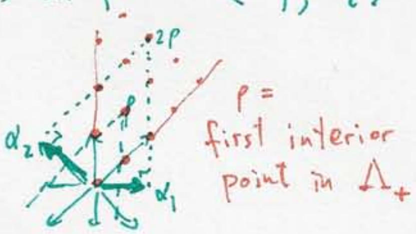
Hopf algebra + R-matrix + charmed element $\Rightarrow$ ribbon category

We have our Hopf algebra : $U_q^{res}(\mathfrak{g})$,

we have the R-matrix $R = q^{\frac{1}{2} H \otimes H} \sum_{m \geq 0} q^{\binom{m}{2}} [m]_q! \, E^{(m)} \otimes F^{(m)}$

Before describing μ (= the charmed element) I need

Def $\rho \in \Lambda_+$ satisfies $\langle 2\rho, \alpha_i \rangle = \langle \alpha_i, \alpha_i \rangle$



$\mu := \prod K_i^{n_i}$ with n_i s.t. $2\rho = \sum n_i \alpha_i$

[in the above example: $\mu = K_1^4 K_2^3$.]

Let's now check that this indeed a charmed element:

- $S^2(a) = \mu a \mu^{-1}$ I'll do it for $a = E_j$

$$E_j \overset{S}{\mapsto} -E_j K_j^{-1} \overset{S}{\mapsto} K_j E_j K_j^{-1} = q^{\langle \alpha_j, \alpha_j \rangle} E_j$$

$$\mu E_j \mu^{-1} = \prod K_i^{n_i} E_j \prod K_i^{-n_i}$$

$$= \prod_i \left(q^{\langle \alpha_i, \alpha_j \rangle} \right)^{n_i} E_j$$

$$= q^{\langle \sum_i n_i \alpha_i, \alpha_j \rangle} E_j = q^{\langle 2\rho, \alpha_j \rangle} E_j = q^{\langle \alpha_j, \alpha_j \rangle} E_j$$

✓

- $S(\mu) = \mu^{-1}$ obvious (by def $S(K_i) = K_i^{-1}$)
- $\Delta(\mu) = \mu \otimes \mu$ obvious ($\Delta(K_i) = K_i \otimes K_i$)
- $m(1 \otimes \mu^{-1}) R \overset{?}{=} m(1 \otimes \mu) \bar{R}$

(we'll only check it for $\mu = K$ $U_q(sl(2))$ since that's the only case for which we have a formula for the R-matrix.)

I'll check this using the formula

$$R = \sum_{n,m} \frac{1}{n!} \left(\frac{1}{2}\log(q)\right)^n q^{\binom{m}{2}} \frac{1}{[m]_q!} H^n E^m \otimes H^n F^m$$

which doesn't make sense in $U_q(sl(2))$ itself, but makes sense in some completion (in particular, completion at ideal $(q-1)$ to make sense of $\log(q)$)

$$\sum_{m,n} \frac{1}{n!} \left(\frac{1}{2}\log(q)\right)^n q^{\binom{m}{2}} \frac{1}{[m]_q!} H^n E^m K^{-1} H^n F^m \quad \leftarrow 1$$

$$\overset{?}{=} \sum_{m,n} \frac{1}{n!} \left(\frac{1}{2}\log(q)\right)^n q^{\binom{m}{2}} \frac{1}{[m]_q!} H^n F^m K H^n E^m \quad \leftarrow 2$$

These elements correspond to the pictures  and 

which are morphisms from X to X

for any $X \in Rep(U_q(\mathfrak{g}))$. In particular they are central: they act by a scalar on each irreducible W_λ.

In order to check if they are equal, it is enough to check that they act by 1 and 2 act the same way on each W_λ.

{ the reason why it's enough is that the actual Hopf algebra we care about is not $U_q(\mathfrak{g})$ but some completion.
1: completion 2: quotient identify things that act the same way on each W_λ. }

So we fix a Weyl module W_λ (we're away from roots of unity so it's irreducible) and want to check that ① and ② act by the same scalar.

To compute the scalar for ① ⇝ check on a lowest weight vector: $v_{-\lambda}$

To compute the scalar for ② ⇝ check on a highest weight vector: v_λ

①: $$\ldots = \sum_n \frac{1}{n!}\left(\tfrac{1}{2}\log(q)\right)^n H^n K^{-1} H^n v_{-\lambda}$$

$$= q^{\frac{1}{2}H^2} K^{-1} v_{-\lambda} = q^{\frac{1}{2}\lambda^2} \cdot q^{\lambda} \cdot v_{-\lambda} = q^{\frac{1}{2}\lambda^2+\lambda} v_{-\lambda}$$

the coefficients agree ☺

②: $$\ldots = \sum_n \frac{1}{n!}\left(\tfrac{1}{2}\log(q)\right)^n H^n K H^n v_\lambda$$

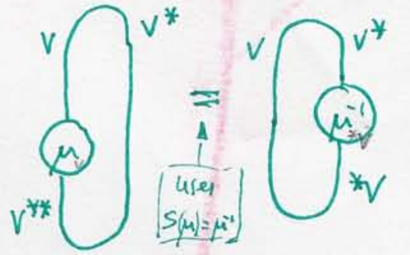
$$= q^{\frac{1}{2}H^2} K v_\lambda = q^{\frac{1}{2}\lambda^2} q^{\lambda} v_\lambda = q^{\frac{1}{2}\lambda^2+\lambda} v_\lambda$$

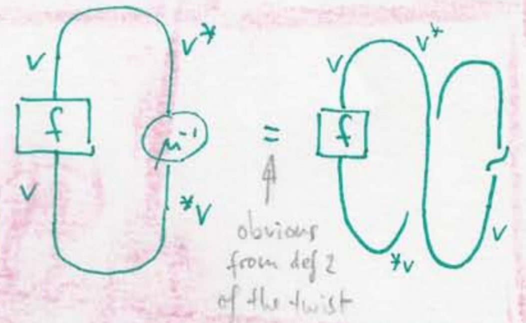
Q.E.D.

As a bonus of the above computation, we learn what the twist is on $\mathrm{Rep}(U_q(\mathfrak{sl}(2)))$

$$\theta_{W_\lambda} = q^{\frac{1}{2}\lambda^2+\lambda} \cdot \mathrm{id}_{W_\lambda} \qquad (\dim W_\lambda = \lambda+1)$$

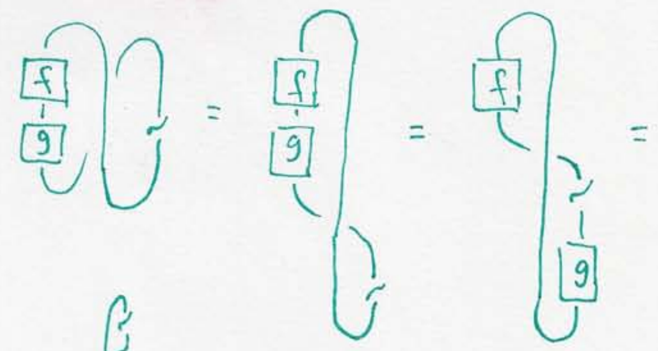
Now that we have a charmed element, we can define a very important concept:

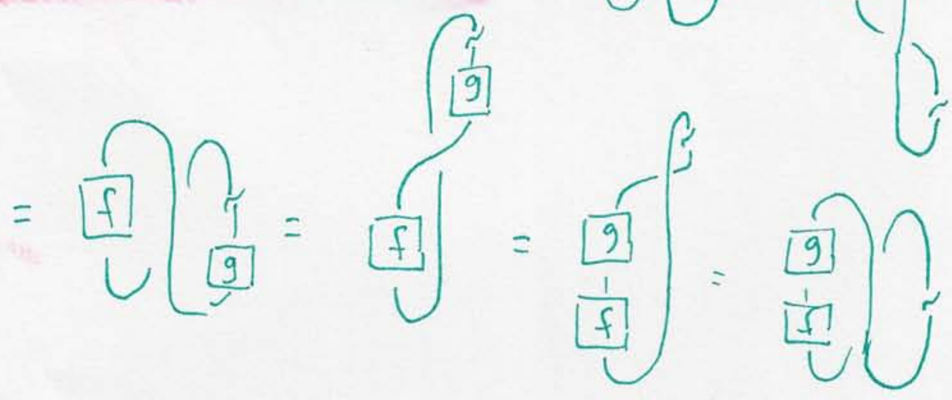
quantum dimension: $\mathrm{qdim}(V) :=$ 

quantum trace: $\mathrm{qtr}(f) :=$ 

main property justifying its name:

$$\boxed{\mathrm{qtr}(fg) = \mathrm{qtr}(gf)}$$:





☒

Def $f : V \to W$ is *negligible* if $\mathrm{qtr}(fg) = 0 \quad \forall\, g : W \to V$.

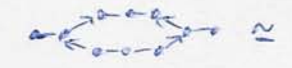
V is *negligible* if ~~$\mathrm{qdim}(V) = 0$~~ id_V is negligible.

(for irreducible modules, this is equivalent to $\mathrm{qdim}(V) = 0$)

I am now in position to define the modular tensor category associated to a quantum group (that's $Z(S^1)$ in Chris' story).

Object: finite dimensional tilting modules for the algebra $U_q^{res}(\mathfrak{g})$ where q is a root of unity (order always divisible by 2)
by 4 if 
by 6 if 
(otherwise it's not always modular)

Morphisms: $\mathrm{Hom}_{U_q^{res}(\mathfrak{g})}(V, W) \Big/ \substack{\text{negligible} \\ \text{morphisms}}$

In that category, it can happen that objects of different dimensions become isomorphic [e.g.  $\simeq 0$] but the quantum dimension is an invariant.

Let's compute some quantum dimensions to get a feeling of which modules are going to die.

$$q\dim(V) = q\mathrm{tr}(1_V) = \mathrm{tr}(\mu_V).$$

for $U_q(SL(2))$ $\mu = K$ and so $q\dim\Big($  $\Big) = [n]_q$

n dots

for q^2 of order ℓ, one then gets $[\ell]_q = 0$ and so W_ℓ is negligible.

The only irreducible object are then

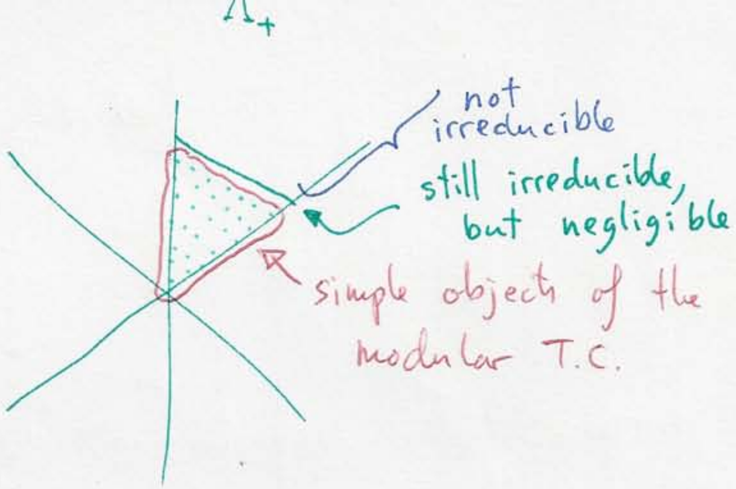
$$W_0 \;,\; W_1 \;,\; \cdots,\; W_{\ell-1}$$

$\leftarrow$ these Weyl modules are irreducible, and hence, they are tilting.

W_ℓ is still irreducible, but it's negligible.

The other Weyl modules are not irreducible.

In general:



I want to finish with a computation that justifies that picture.

Recall that the character of a module M is given by

$$\cancel{\big(\chi(M)\big)(t) = \sum_{\mu\in\Lambda} \dim(M_\mu)\, e^{\langle\mu,t\rangle}}$$

redef: $\boxed{\big(\chi(M)\big)(\tau) = \sum_{\mu\in\Lambda} \dim(M_\lambda)\, q^{\langle\mu,\tau\rangle} \in \mathrm{Fun}\big(\Lambda, \mathbb{Z}[q,q^{-1}]\big)}$

$$= \mathrm{tr}_M(q^\tau)$$

In particular, $\quad q\dim(M) = \mathrm{tr}_M(q^{2\rho}) = \chi(M)(2\rho)$

Recall the Weyl character formula from lecture one:

$$\chi(W_\lambda) = \frac{\sum_{w\in W} (-1)^w q^{w(\lambda+\rho)}}{\sum_{w\in W} (-1)^w q^{w\rho}}$$

before I wrote it using the shifted Weyl group action, but one can trade that for a "$+\rho$".

along with the formula $\sum_{w\in W} (-1)^w q^{w\rho} = \prod_{\alpha\in\Delta_+} \left(q^{-\frac{\alpha}{2}} - q^{\frac{\alpha}{2}}\right)$ for the denominator.

We then have

$$\boxed{q\dim(W_\lambda)} = \chi(W_\lambda)(2\rho) = \frac{\sum (-1)^w q^{\langle w(\lambda+\rho), 2\rho\rangle}}{\sum (-1)^w q^{\langle w\rho, 2\rho\rangle}}$$

$$= \frac{\prod_{\alpha\in\Delta_+} q^{\langle\alpha,\lambda+\rho\rangle} - q^{-\langle\alpha,\lambda+\rho\rangle}}{\prod_{\alpha\in\Delta_+} q^{\langle\alpha,\rho\rangle} - q^{-\langle\alpha,\rho\rangle}} = \boxed{\prod_{\alpha\in\Delta_+} \frac{[\langle\alpha,\lambda+\rho\rangle]_q}{[\langle\alpha,\rho\rangle]_q}}$$

For $q^{2\ell} = 1$, this expression vanishes when $\ell \mid \langle\alpha,\lambda+\rho\rangle$ for some $\alpha\in\Delta_+$. The first occurrence is $\alpha = \alpha_0 = \text{highest root}$, and $\boxed{\langle\alpha_0, \lambda+\rho\rangle = \ell}$.

That equation defines a hyperplane in Λ, and that's the one that defines the Weyl alcove.