Quantum groups seminar

Talk 1: Introduction

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Talk by André, notes by Ralph Klaasse[1], 3 pages

Usually the discussion of a new mathematical object starts with its definition and some basic properties. However, in this case a straightforward definition of a quantum group is hard to give. Nevertheless, we can at this point say the following: a quantum group is a special type of Hopf algebra, namely a deformation of a Lie group or a Lie algebra.

Recall the Lie algebra $\mathfrak{sl}(2)$. A basic description is the set of 2x2-matrices with zero trace, i.e.

$$\mathfrak{sl}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$

The Lie bracket is the standard commutator bracket, $[A, B] = AB - BA$. A more useful point of view is to consider $\mathfrak{sl}(2) = \operatorname{span}\{E, F, H\}$, where E, F and H are given by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is actually not that important what E, F and H look like explicitly; the most important thing to remember is their Lie brackets:

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

It is well-known that for every positive integer n, there is exactly one n-dimensional irreducible representation of $\mathfrak{sl}(2)$.

Picture of this representation when $n = 5$: write five dots in a row for the vector space's basis elements. E shifts to the right and multiplies by 1, 2, 3 and 4 respectively, while F shifts to the left and multiplies by 4, 3, 2 and 1 respectively. H merely multiplies by -4, -2, 0, 2 and 4 respectively.

One can check that the commutation relations hold. For example, considering the relation $[E, F] = H$ at the fourth basis element we see that

$$[E, F] = EF - FE = 2 \cdot 3 - 1 \cdot 4 = 2 = H,$$

as required.

We will now consider something called Quantum $\mathfrak{sl}(2)$. The idea is to introduce a formal variable q used to deform $\mathfrak{sl}(2)$. As $q \to 1$, one should recover the "classical situation", i.e. that without deformation. Let $n \in \mathbb{N}$ be given. Then $[n]_q$ denotes the q-quantum analogon of n. It is defined by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+3} + \ldots + q^{n-3} + q^{n-1}.$$

[1] Any mistakes or inaccuracies are very likely to be mine, not André's.

The sum on the right has n terms, so that indeed as $q \to 1$ we get $[n]_q \to n$. Some examples of $[n]_q$ can be found below.

$$\begin{aligned}
n = 1: &\qquad [1]_q = 1, \\
n = 2: &\qquad [2]_q = q^{-1} + q, \\
n = 3: &\qquad [3]_q = q^{-2} + 1 + q^2, \\
n = 4: &\qquad [4]_q = q^{-3} + q^{-1} + q + q^3.
\end{aligned}$$

Now, to get (a representation of) Quantum $\mathfrak{sl}(2)$, one again has similar operators E and F, but uses a new operator $K = q^H$ instead of H. This is due to the fact that one wants algebraic bracket relations between these operators. To get the information of H back from K, one has to take its "derivative" in the direction of q. At any rate, in the $n = 5$ example, one merely replaces every natural number used for E and F by its q-quantum analogon. K now merely multiplies by q^{-4}, q^{-2}, 1, q^2 and q^4 respectively. Do similar Lie bracket relations hold? Indeed, we have:

Exercise. *Let $m, n \in \mathbb{N}$ be given. Show that $[n]_q[m]_q - [n-1]_q[m+1]_q = [m-n+1]_q$.*

One can furthermore check that $[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$, or just $[H]_q$, but this is not an algebraic relation. Note that one can take the vector space of the representation as merely over $\mathbb{C}$, or $\mathbb{C}(q)$ if q is considered to be formal. Another choice is to just use $\mathbb{Z}[q, q^{-1}]$. If one wants to consider q as lying in a formal neighborhood of 1, a fourth choice is to use the parameter h given by $q = e^h$ and use $\mathbb{C}[[h]]$.

In this context, quantum means we are dealing with commutative spaces which are replaced by non-commutative spaces through deformation. A general idea of non-commutative geometry is that a space X should contain exactly as much information as its algebra of functions $X \to \mathbb{C}$. The functions one considers depends on the context: when X is a topological space, one uses continuous functions, when X is an algebraic variety, one uses algebraic functions, et cetera. The steps one takes can roughly be described as follows: take a space X, take its commutative algebra of functions $X \to \mathbb{C}$ and then lose commutativity to get merely an associative algebra.

Indeed, given a Lie group G, we get an associative algebra (A, μ, η) where μ comes from the multiplication on G and η denotes evaluation at the unit. The structure of G gives rise to the following maps through pullback:

$$\begin{array}{rcll}
G & \rightsquigarrow & (A, \mu, \eta), & \\
m : G \times G \to G & \rightsquigarrow & \Delta : A \to A \otimes A & \text{coproduct}, \\
e : \{*\} \to G & \rightsquigarrow & \varepsilon : A \to \mathbb{C} & \text{counit}, \\
(\cdot)^{-1} : G \to G & \rightsquigarrow & s : A \to A & \text{antipode}.
\end{array}$$

Here and beyond we tacitly assume the base field is $\mathbb{C}$. This leads us to consider the following algebraic structures, called Hopf algebras.

Definition. *A Hopf algebra is a vector space A with an associative product μ, a unit η, a co-associative product Δ, a counit ε and an antipode s (satisfying various axioms).*

Here we have

$$\begin{aligned}
\mu &: A \otimes A \to A, \\
\eta &: \mathbb{C} \to A, \\
\Delta &: A \to A \otimes A, \\
\varepsilon &: A \to \mathbb{C}, \\
s &: A \to A.
\end{aligned}$$

Note the symmetry between the axioms: if A is a Hopf algebra, then its dual A^* is as well (symmetry between μ, η and Δ, ε).

An example of a Hopf algebra is the following: let $\mathfrak{g}$ be a (simple) Lie algebra. Then $U(\mathfrak{g})$, the universal enveloping associative algebra of $\mathfrak{g}$, is a Hopf algebra. As the above maps are homomorphisms, it suffices to describe them on generators $x \in \mathfrak{g}$ of $\mathfrak{g}$. We have $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, $s(x) = -x$.

This then leads to $U_q(\mathfrak{g})$, the quantum groups as 1-parameter deformations in the moduli space of Hopf algebras. An alternative way is to let G be an algebraic group, consider $\mathbb{C}[G]$, the space of algebraic functions $G \to \mathbb{C}$ (which is an commutative or associative algebra), and then obtain a commutative Hopf algebra. In fact, this gives the dual of $U_q(\mathfrak{g})$, when $\mathfrak{g}$ is G's Lie algebra.