Quantum groups seminar

Talk 2: Preliminaries
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Talk by Ralph Klaasse, 4 pages

These are notes to a talk given in the quantum groups seminar during the fall of 2012 at Utrecht University, which followed the book Quantum Groups by Kassel.

1 Introduction

In this talk we start building the theory of quantum groups in earnest. As was mentioned in the previous talk, they are special types of Hopf algebras that arise as deformations of Lie groups or Lie algebras. To describe how this process works, we will need to first develop some theory on algebras and their modules. In particular, we will discuss the so-called Ore extension of an algebra. Later we will see that enveloping algebras of solvable Lie algebras are Ore extensions, as are the coordinate rings of quantum groups. Ore extensions will be also be used when we see that certain objects of interest can be realized as iterated Ore extensions, which in turn is used to show these objects have certain properties preserved by Ore extension (see e.g. [Kassel, VI.1.4] on $U_q(\mathfrak{sl}(2))$).

Throughout this talk I will assume familiarity with the following concepts: (free) algebras, modules, matrix algebras, graded algebras and Noetherian rings. These can all be found in [Kassel, I.1-I.6 and I.8].

2 Ore extensions

Let R be an algebra over a field k and let $R[t]$ be the free left R-module of all polynomials $P \in R[t]$ of the form

$$P = a_n t^n + \cdots + a_0 t^0,$$

where the coefficients a_i lie in R. When $a_n \neq 0$ we define the degree $\deg(P)$ of P as being equal to n; we set $\deg(0) = -\infty$. We wish to turn $R[t]$ into an algebra, such that its structure is compatible with that on R and the degree. In fact, we wish to find all such structures. It is clear that we must specify or describe what the product ta is for arbitrary elements $a \in R$.

Let α be an algebra endomorphism of R.

Definition. *An α-derivation of R is a linear endomorphism δ of R such that for all $a, b \in R$*

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b.$$

Note that we have $\delta(1) = 0$, as

$$\delta(1) = \delta(1 \cdot 1) = \alpha(1)\delta(1) + \delta(1)1 = \alpha(1)\delta(1) + \delta(1),$$

but $\alpha(1) = 1$ as α must preserve units. We will see that these two pieces of data (i.e., a pair (α, δ)) characterize all possible algebra structures on $R[t]$. We first show that given a compatible algebra structure on $R[t]$ we can extract a pair (α, δ) from it.

Theorem [Kassel, I.7.1.(a)]. *Suppose $R[t]$ is given an algebra structure such that:*

- *the natural inclusion $R \hookrightarrow R[t]$ is an algebra morphism;*
- $\deg(PQ) = \deg(P) + \deg(Q)$ *for all $P, Q \in R[t]$.*

Then R has no zero divisors, and there exists a unique injective algebra endomorphism $\alpha : R \to R$ and a unique α-derivation δ of R such that $ta = \alpha(a)t + \delta(a)$ for all $a \in R$.

Proof We first show R has no zero divisors. Let $a, b \in R$ be nonzero elements. Then through the natural inclusion they are of degree 0 in $R[t]$. Through compatibility we have $\deg(ab) = \deg(a) + \deg(b) = 0$, so that we must have $ab \neq 0$ in $R[t]$, and hence in R.

We now show existence and uniqueness of α and δ. Let $a \in R$ be any nonzero element and consider it in $R[t]$. Then we have $\deg(ta) = \deg(t) + \deg(a) = 1$, so that by definition of $R[t]$ there are unique elements $b, c \in R$ such that

$$ta = bt + c.$$

In fact, we know that b must be nonzero. Now define α and δ uniquely through $\alpha(a) := b$ and $\delta(a) := c$. As left multiplication by t is linear, i.e. $t(\lambda a + a') = \lambda ta + ta'$ for all $\lambda \in k$ and $a, a' \in R$, we see that so are α and δ:

$$\begin{aligned}\alpha(a + a')t + \delta(a + a') = t(a + a') = ta + ta' = \alpha(a)t + \delta(a) + \alpha(a') + \delta(a');\\ \alpha(\lambda a)t + \delta(\lambda a) = t(\lambda a) = \lambda ta = \lambda(\alpha(a)t + \delta(a)).\end{aligned}$$

Furthermore, as we saw that $\alpha(a)$ is nonzero for nonzero a, we see that α is injective. Now take two elements $a, b \in R$ and consider the relation $(ta)b = t(ab)$ in associative algebra $R[t]$. We get

$$\alpha(a)(\alpha(b)t + \delta(b)) + \delta(a)b = (\alpha(a)t + \delta(a))b = (ta)b = t(ab) = \alpha(ab)t + \delta(ab).$$

Looking at this per degree we see that

$$\begin{aligned}\alpha(ab) &= \alpha(a)\alpha(b);\\ \delta(ab) &= \alpha(a)\delta(b) + \delta(a)b.\end{aligned}$$

Furthermore, using that $t1 = t$ we see from $ta = \alpha(a)t + \delta(a)$ that $\alpha(1) = 1$ and $\delta(1) = 0$. We conclude that α is an injective algebra endomorphism and that δ is an α-derivation. ■

This theorem in fact has a converse, in that a pair (α, δ) is enough to uniquely specify a compatible algebra structure on $R[t]$.

Theorem [Kassel, I.7.1.(b)]. *Let R be an algebra without zero divisors. Given an injective algebra endomorphism α of R and an α-derivation δ of R there exists a unique algebra structure on $R[t]$ such that:*

- *the natural inclusion $R \hookrightarrow R[t]$ is an algebra morphism;*
- *$ta = \alpha(a)t + \delta(a)$ for all $a \in R$.*

When $R[t]$ is given this algebra structure it is called the *Ore extension* $R[t, \alpha, \delta]$ attached to the data (R, α, δ). There are several special cases covered by this object:

- If $\alpha = \mathrm{id}_R$ and $\delta = 0$, then $R[t, \mathrm{id}_R, 0] = R[t]$, where now t commutes with all elements of R;
- Somewhat more generally, if $\alpha = \mathrm{id}_R$, then $R[t, \mathrm{id}_R, \delta]$ is an algebra of polynomial differential operators, satisfying the Leibniz formula.

Exercise [Kassel, I.9.8]. *Let δ be an α-derivation of an algebra R. Prove that if $a_1, \dots, a_n$ are elements of R, then*

$$\delta(a_1 \cdot \ldots \cdot a_n) = \delta(a_1)a_2 \cdot \ldots \cdot a_n + \sum_{i=2}^{n-1} \alpha(a_1 \cdot \ldots \cdot a_{i-1})\delta(a_i)a_{i+1} \cdot \ldots \cdot a_n + \alpha(a_1 \cdot \ldots \cdot a_{n-1})\delta(a_n),$$

and for $n \geq 1$

$$\delta^n(a_1 a_2) = \sum_{k=0}^{n} S_{n,k}(a_1)\delta^{n-k}(a_2),$$

where the $S_{n,k}$ is the linear endomorphism of R defined as the sum of all $\binom{n}{k}$ compositions of k copies of δ and $n-k$ copies of α.

Proof As was noted before, it clearly suffices to prescribe the product ta for all $a \in R$ to determine the product on $R[t]$, by associativity. As such, the relation $ta = \alpha(a)t + \delta(a)$ specifies the algebra structure on $R[t]$ uniquely.

We now show this structure actually exists. Let M denote the associative algebra of all infinite matrices $(f_{ij})_{i,j\geq 1}$ with entries in the algebra $\mathrm{End}(R)$ such that each row and column has only finitely many nonzero entries. Let the unit of M be denoted by I, and given an element $a \in R$, let $\widehat{a} \in \mathrm{End}(R)$ denote left multiplication by a. We then have

$$\begin{aligned}\alpha\widehat{a} &= \widehat{\alpha(a)}\alpha;\\ \delta\widehat{a} &= \widehat{\alpha(a)}\delta + \widehat{\delta(a)}.\end{aligned}$$

Now let $T \in M$ be the infinite matrix defined by

$$T := \begin{pmatrix} \delta & 0 & 0 & 0 & \cdots \\ \alpha & \delta & 0 & 0 & \cdots \\ 0 & \alpha & \delta & 0 & \cdots \\ 0 & 0 & \alpha & \delta & \cdots \\ 0 & 0 & 0 & \alpha & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then define the linear map $\Phi : R[t] \to M$ by

$$\Phi\left(\sum_{i=0}^{n} a_i t^i\right) = \sum_{i=0}^{n} (\widehat{a}_i I)\, T^i.$$

We assert that Φ is injective. If e_i is the column vector which is zero everywhere except at the ith entry, where it is equal to the unit 1 of R, we see that $T(e_i) = e_{i+1}$ for all $i \geq 1$. Now let $P \in R[t]$ be given such that $\Phi(P) = 0$. Then we have

$$0 = \Phi(P)(e_1) = \sum_{i=1}^{n} (\widehat{a}_i I)\, T^i(e_1) = \sum_{i=0}^{n} \widehat{a}_i e_{i+1}.$$

From this we clearly see that $\widehat{a}_i = 0$ for all i, and applying each to $1 \in R$ we get $a_i = 0$ for all i. Hence $P = 0$ and Φ is injective. Now note that the following relation holds:

$$T(\widehat{a}I) = \left(\widehat{\alpha(a)}I\right)T + \left(\widehat{\delta(a)}I\right).$$

Let S be the subalgebra of M generated by T and elements $\widehat{a}I$ for all $a \in R$. Then by this relation we see that S is the image of $R[t]$ under Φ. As this map is injective, it in fact induces a linear isomorphism from $R[t]$ to the algebra S. With this we an lift the algebra structure from S to $R[t]$, showing existence of the desired algebra structure on $R[t]$. ■

Note that the Ore extension $R[t,\alpha,\delta]$ has no zero divisors by the existence of the degree. It is free as a left R-module, with basis $\{t^i\}_{i\geq 0}$. If α is assumed to be an automorphism (i.e. it is invertible) we also have the following:

Corollary. *If α is an automorphism, $R[t,\alpha,\delta]$ is a free right R-module with basis $\{t^i\}_{i\geq 0}$.*

Proof We prove $\{t^i\}_{i\geq 0}$ generates $R[t,\alpha,\delta]$ as right R-module. We wish to show each element P of $R[t,\alpha,\delta]$ can be written as $P = \sum_{i=0}^{n} t^i a^i$ for some n and $a_0,\ldots,a_n \in R$. We prove this through induction on the degree n of P. When $n = 0$ it is clear, whereas for higher n we have

$$at^n = t^n\alpha^{-n}(a) + \text{lower degree terms}.$$

Now suppose $\{t^i\}_{i\geq 0}$ is not free. Then there exists a relation

$$t^n a_n + t^{n-1}a_{n-1} + \cdots + ta_1 + a_0 = 0,$$

where $a_n \neq 0$. By the above relation we can rewrite this as a relation

$$\alpha^n(a_n)t^n + \text{lower degree terms} = 0,$$

so that $\alpha^n(a_n) = 0$. As α is an isomorphism we get $a_n = 0$, which is a contradiction. ■

3 Noetherian rings

Recall that a ring A is called left Noetherian if any left ideal I of A is finitely generated, or equivalently if any ascending sequence of left ideals of A is finite. We know that being left Noetherian is preserved by surjective ring morphisms, but also by Ore extension.

Theorem [Kassel, I.8.3]. *Let R be an algebra without zero divisors, α an R-algebra automorphism and δ an α-derivation of R. If R is left Noetherian, then so is $R[t, \alpha, \delta]$.*

Proof Let I be a left ideal of $R[t, \alpha, \delta]$. We wish to prove I is finitely generated. Now, given an integer $d \geq 0$ define

$$I_d = \{0\} \cup \{a \in R \,|\, a \text{ is the leading coeffiient of a degree } d \text{ element of } I.\}\,.$$

We assert I_d is a left ideal of R. Clearly it is closed under addition. Now assume $0 \neq a \in I_d$ so that $at^d + \cdots \in I$. Given any nonzero $b \in R$ we then have $bat^d + \cdots \in I$ as I is an ideal. Because R has no zero divisors we have $ba \neq 0$ so that $ba \in I_d$. Hence I_d is a left ideal.

Now note that if $a \in I_d$ is the leading coefficient of some polynomial P, then $tP = \alpha(a)t^{d+1} + \ldots$, so that $\alpha(a)$ is the leading coefficient of tP. This clearly implies $\alpha(I_d) \subset I_{d+1}$. With this we can form the following ascending sequence of left ideal in R:

$$I_0 \subset \alpha^{-1}(I_1) \subset \alpha^{-2}(I_2) \subset \ldots \subset \alpha^{-n}(I_n) \subset \ldots.$$

As R is left Noetherian, there exists an integer n such that $\alpha^{-n}(I_n) = \alpha^{-(n+i)}(I_{n+i})$, or $I_{n+i} = \alpha^i(I_n)$ for all $i \geq 0$. As R is left Noetherian any left ideal is finitely generated, so for d with $0 \leq d \leq n$, choose generators $a_{d,1}, \ldots, a_{d,p}$ of I_d. For $1 \leq i \leq p$, let $P_{d,i}$ be any degree d polynomial whose leading coefficient is $a_{d,i}$. Then the collection $\{P_{d,i}\}$ is finite.

Let I' be the left ideal of $R[t, \alpha, \delta]$ generated by this family. Obviously we have $I' \subset I$. We claim the converse inclusion also holds by induction on degrees. Let $P \in I$ be any element. If $\deg(P) = 0$, then clearly we have $P \in I'$. So now suppose that $Q \in I'$ for all $Q \in I$ with $\deg(Q) < d$ and let $P \in I$ be of degree d. There now are two cases:

(1) If $d \leq n$, then the leading coefficient a of P is of the form $a = \sum_{i=0}^{p} r_i a_{d,i}$ for some $r_i \in R$. Hence $Q := P - \sum_{i=0}^{p} r_i P_{d,i}$ is an element of I with degree less than d. By induction we have $Q \in I$ and hence $P \in I$.

(2) If $d > n$, then the leading coefficient a of P belongs to $I_d = \alpha^{d-n}(I_n)$. As such it can be written as $a = \sum_{i=0}^{p} r_i \alpha^{d-n}(a_{d,i})$ for some $r_i \in R$. Now consider $Q := P - \sum_{i=0}^{p} r_i t^{d-n} P_{d,i}$. The coefficient of t^d in Q is equal to $a - \sum_{i=0}^{p} r_i \alpha^{d-n}(a_{d,i}) = 0$, so that $\deg(Q) < d$. By induction we have $Q \in I$ and hence $P \in I$.

■