Quantum groups, talk 1

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1 SL(2) and the ordinary plane

1.1 In terms of spaces

Recall the construction of the matrix groups GL_2 and SL_2 acting on the plane:

- We define the set of all 2×2 matrices by

$$M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{A}^4 \right\} \simeq \mathbb{A}^4$$

  There is a multiplication and a unit

$$\mu : M_2 \times M_2 \longrightarrow M_2 \qquad \eta : * \longrightarrow M_2$$

  As functions from/to $\mathbb{A}^4$, these are polynomials in terms of the coordinates $a, ..., d$. There is also a function det on M_2, equal to $ad - bc$ in coordinates, so also a polynomial.

- We can define the space of invertible matrices by

$$GL_2 = \{(A, t) \in M_2 \times \mathbb{A} \big| \det(A)t - 1 = 0\}$$

  With inversion:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t) \mapsto (\frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \frac{1}{t}) = (t \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad - bc)$$

- $$SL_2 = \{A \in M_2 \big| \det(A) - 1 = 0\}$$

- Action on plane

$$M_2 \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

1.2 In terms of algebras

There is a functor

$$\text{Spaces}_k^{op} \longrightarrow \text{comm. k-Alg}$$

which takes a space X to the algebra of global (regular) functions on X. We have

- in all previous diagrams, the arrows reverse
- products of spaces corresponds to coproduct (=tensor product) of commutative k-algebras
- zero sets of functions correspond to quotients of k-algebras by these functions.

For example, we have that $\mathbb{A}^n \leftrightarrow k[x_1, ..., x_n]$. We interpret each of the x_i as the i-th coordinate function.

Then:

$$M(2) \simeq k[a, b, c, d] \qquad GL(2) \simeq k[a, b, c, d, t]/((ad-bc)t-1) \qquad SL(2) = k[a, b, c, d]/(ad-bc-1)$$

Note that the determinant is a function on M_2, so it is an element of $M(2)$ (nl. $ad - bc$).

Now, multiplication induces a map $\Delta : M(2) = k[a,b,c,d] \longrightarrow M(2)^{\otimes 2} = k[a,..,d] \otimes k[a,..,d]$ given by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta a & \Delta b \\ \Delta c & \Delta d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$$

The unit gives a map $\epsilon : M(2) = k[a,b,c,d] \longrightarrow k$ given by

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All maps are extended to be algebra maps.

For $GL(2)$, we extend the above as:

$$\begin{aligned} \Delta(t) &= t \otimes t && \text{(rest as for } M(2)) \\ \epsilon(t) &= 1 && \text{(rest as for } M(2)) \\ s(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t) &= (\frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \frac{1}{t}) \\ &= (t \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad - bc) && \text{extend as antihom.} \end{aligned}$$

For $SL(2)$ it is just the maps for $M(2)$, together with inversion

$$s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{extend as antihom.}$$

We have to check that the above maps respect the relations we impose on $GL(2), SL(2)$, e.g. $\Delta(t) \cdot \Delta(ad - bc) = 1 \otimes 1$ etc. This follows from the facts that

$$\Delta(\det) = \Delta(ad - bc) = \det \otimes \det \qquad \epsilon(\det) = \epsilon(ad - bc) = 1$$

which basically say that the determinant is multiplicative and $\det(id) = 1$.

Proposition 1.1. *$M(2)$ is a bialgebra and $GL(2)$, $SL(2)$ are Hopf algebras.*

Proof. Note that the coalgebra maps are maps of algebras by construction, so we only have to check that Δ, ϵ give a coalgebra. This is basically the fact that matrix multiplication is associative (as well as taking tensor products) and has id as its unit (and that 1 is also the unit element for tensoring with k). For example, we have that

$$(1 \otimes \epsilon)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \otimes \epsilon \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

□

Action of e.g. $GL(2)$ on the plane: map $GL_2 \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ so that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Dually, this action determines a map $\rho : k[x', y'] \longrightarrow GL(2) \otimes k[x,y]$ given by

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \otimes x + b \otimes y \\ c \otimes x + d \otimes y \end{pmatrix}.$$

Proposition 1.2. *ρ establishes $k[x,y]$ as a left $M(2)/GL(2)/SL(2)$-comodule.*

Proof. Similar proof as for showing $GL(2)$ to be a Hopf algebra: essentially this is just matrix multiplication. □

Definition 1. *Let H be a bialgebra, A an algebra and $A \longrightarrow H \otimes A$ an algebra homomorphism. Then A is a left H comodule if the following diagrams commute*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\rho"'] & H \otimes A \arrow[d, "1 \otimes \rho"] \\ H \otimes A \arrow[r, "\Delta \otimes 1"'] & H \otimes H \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr, "1"'] & H \otimes A \arrow[d, "\epsilon \otimes 1"] \\ & A \end{tikzcd}$$

2 Quantum plane and quantum groups

2.1 Quantum plane

Quantum plane

$$k_q[x,y] = k\{x,y\}/(yx - qxy) = k < x, y, xy, yx, ... > /(yx - qxy).$$

Rules for doing computations in this algebra:

- $y^j x^i = q^{ij} x^i y^j$
- Set $(n_q) = \frac{q^n - 1}{q-1} = 1 + q + ... + q^{n-1}$ and $(n)!_q = (n)_q ... (2)_q (1)_q$ for $n > 0$. Finally, set

$$\binom{n}{k}_q = \frac{(n)!_q}{(k)!_q (n-k)!_q}.$$

  Then $\binom{n}{k}_q = \binom{n}{n-k}_q$ and we have the q-Pascal identity

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q.$$

- Binomial expansion:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$$

2.2 $M_q(2)$

From now on, assume $q^2 \neq -1$!!!!

- 4 generators
- action on quantum plane

$$\rho : k_q[x,y] \longrightarrow M_q(2) \otimes k_q[x,y] \qquad \rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \otimes x + b \otimes y \\ c \otimes x + d \otimes y \end{pmatrix}$$

  is again an element of the quantum plane.

- transposes

$$\rho^T : k_q[x,y] \longrightarrow M_q(2) \otimes k_q[x,y] \qquad \rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \otimes x + c \otimes y \\ b \otimes x + d \otimes y \end{pmatrix}$$

Proposition 2.1. *The k-algebra that has these properties is generated by a, b, c, d subject to the following 6 relations*

$$\begin{aligned} ba &= qab & db &= qbd \\ ca &= qac & dc &= qcd \\ bc &= cb & ad - da &= (q^{-1} - q)bc \end{aligned}$$

We denote it $M_q(2)$.

Proof.

$$\rho(qx'y') = \rho(y'x') \qquad \Rightarrow \qquad q(a\otimes x + b\otimes y)(c\otimes x + d\otimes y) = (c\otimes x + d\otimes y)(a\otimes x + b\otimes y).$$

Comparing the x^2-, xy- and y^2-terms gives

$$qac = ca \qquad qbd = db \qquad q^2bc + qad = qda + cb$$

For ρ^T we find:

$$qab = ba \qquad qcd = dc \qquad qad + q^2cb = qda + bc$$

Subtraction gives that $(q^2+1)bc = (q^2+1)cb$, so $bc = cb$ and then the last relation follows. □

As an algebra, it is Noetherian (not sure if that is important).

Our quantum matrices form a 'quantum monoid', i.e. a bialgebra:

Proposition 2.2. *$M_q(2)$ forms a bialgebra under the familiar operations*

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\epsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. Note that the coalgebra maps are maps of algebras by construction, so we only have to check that Δ and ϵ give a coalgebra structure. This is basically the fact that matrix multiplication is associative (as well as taking tensor products) and has id as its unit (and that 1 is also the unit element for tensoring with k).

So it suffices to check that Δ and ϵ are actually well-defined, i.e. that they respect the six relations we have imposed on a, b, c, d in the construction of $M_q(2)$. This can be done by a bunch of tedious computations.

Maybe more conceptual: the proof of the previous proposition shows the following: if x, y are two elements of *any* ring such that $yx = qxy$, and we have *any* matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

satisfy $y'x' = qx'y'$ (and the same with the transpose of the above matrix), then a, b, c, d satisfy the six relations of proposition 2.1. We apply this to the matrix $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: we have that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \rho(x) \\ \rho(y) \end{pmatrix}$$

satisfy the quantum plane relation $y'x' = qx'y'$: indeed, by definition of $M_q(2)$ these relations hold for $\rho(x), \rho(y)$, so they will also hold if we multiply by a matrix in $M_q(2)$ one more time. But this means that $\Delta(a), \Delta(b), \ldots$ satisfy the above 6 relations, so we see that Δ indeed preserves the six relations imposed in the construction of $M_q(2)$.

For ϵ a computation immediately shows that it preserves all 6 relations. □

Maybe should talk about R-points, that simplifies some of the proofs?

2.3 $SL_q(2)$ and $GL_q(2)$

We define $SL_q(2)$ and $GL_q(2)$ similar to the classical case: first, we need a quantum determinant

Definition 2. *The quantum determinant is given by*

$$\det{}_q = ad - q^{-1}bc = da - qbc \in M_q(2).$$

It is a central element.

To some extend this is a canonical element: if q not a root of unity, then the center of $M_q(2)$ is generated by $\det_q$.

We define the quantum groups of matrices as before:

$$GL_q(2) = M_q(2) \otimes k[t]/(\det{}_q t - 1)$$
$$SL_q(2) = M_q(2)/(\det{}_q - 1)$$

and we also define Δ and ϵ as in the classical case. Furthermore set

$$s(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t) = (t \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}, \det{}_q) \qquad GL_q(2)$$
$$s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \qquad SL_q(2)$$

Proposition 2.3. *$GL_q(2)$ and $SL_q(2)$ form a Hopf algebra under these maps.*

Exercise: Show that the quantum determinant is multiplicative, i.e.

$$\Delta(\det{}_q) = \det{}_q \otimes \det{}_q \qquad \epsilon(\det{}_q) = 1.$$

Using this and the fact that Δ and ϵ are well defined on M_2, show that Δ and ϵ are well defined on $SL_q(2)$ and $GL_q(2)$.

Proof. It is easily checked that Δ and ϵ give a bialgebra structure on $GL_q(2), SL_q(2)$ (as for $M_q(2)$). A computation shows that s is a well-defined map.

To check that s is an antipode (e.g. for $SL_q(2)$), we have

$$\begin{aligned}
\mu(s \otimes 1)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mu \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \mu \begin{pmatrix} d \otimes a - qb \otimes c & d \otimes b - qb \otimes d \\ -q^{-1}c \otimes a + a \otimes c & -q^{-1}c \otimes b + a \otimes d \end{pmatrix} \\
&= \begin{pmatrix} \det_q & 0 \\ 0 & \det_q \end{pmatrix} \\
&= \eta\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

□

So we have Hopf-algebras, i.e. quantum groups. Note that these Hopf-algebras are not as trivial as their classical analogues: for instance, for $SL_q(2)$, we have that

$$s^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & q^2 b \\ q^2 c & d \end{pmatrix}$$

so whenever $q^2 \neq 1$, applying 'inversion' twice will not give the identity.

Finally, we have that $M_q(2), GL_q(2)$ and $SL_q(2)$ act on the quantum plane: we just set

$$\rho : k_q[x, y] \longrightarrow M_q(2) \otimes k_q[x, y] \qquad \rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \otimes x + b \otimes y \\ c \otimes x + d \otimes y \end{pmatrix}$$

By our assumption on $M_q(2)$, this is a ring homomorphism. Moreover, we have that

$$(\Delta \otimes 1) \circ \rho = (1 \otimes \rho) \circ \rho : k_q[x, y] \longrightarrow M_q(2) \otimes M_q(2) \otimes k_q[x, y]$$

and

$$(\epsilon \otimes 1)\rho = 1 : k_q[x, y] \longrightarrow k_q[x, y]$$

(which again follows pretty much from matrix multiplication).

But now $SL_q(2)$ is a quotient algebra of $M_q(2)$ and the quotient repsects the coalgebra structure. Hence

Proposition 2.4. *The above co-action of $M_q(2)$ on the quantum plane restricts to an action of $SL_q(2)$ on the quantum plane, given by the same formulas.*

Also, $GL_q(2)$ acts naturally on the quantum plane.