Quantum Affine to XXZ/6-Vertex Model

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5 december 2013

Contents

1 Introduction

We have seen in a previous lecture that by taking the braid limit on the $RTT = TTR$ equation, we found the defining relations of the $U_q Sl_2$ algebra. The program that has been set out after these findings is to turn the story around. We start with some algebra and try to solve different models. For example, the Yangian of Sl_2 solves the XXX-model as we have seen.

Model	Coupling	Algebra
XXX	$g_x = g_y = g_z$	Sl_2 Yangian
XXZ	$g_x = g_y$	Quantum Affine Sl_2
XYZ	$g_x \neq g_y \neq g_z$	Elliptic Sl_2

The algebra used to solve the different models is presented in the table above. In these lecture notes we will solve the XXZ-model using the quantum affine algebra of Sl_2. First we will construct the quantum affine algebra using Dynkin diagrams (and loop algebras). The next thing we will see is the universal R-matrix as found by Drinfeld for a general Lie group. After these general discussions about quantum affine groups we focus on the group Sl_2 and explicitly compute the R-matrix for the XXX-model.

We have seen in a previous lecture that the Yang-Baxter equation implies the commutativity of transfer matrices. A feature of an integrable model is the existence of an infinite number of conservation laws. For the XXZ-model there exists a hierarchy of independent operators $H_1, H_2, ...H_n$, including $H_{XXZ} = H_1$, such that $[H_n, H_m] = 0$, $\forall$ m, n.

For the 6-vertex model there exists a family of transfer matrices $T(\xi)$, such that $[T(\xi), T(\xi')] = 0$, $\forall\ \xi, \xi'$. Additionally, the 6-vertex model and the XXZ-model are connected through the following relation

$$H_n = c(\xi \frac{d}{d\xi})^n \log T(\xi))|_{\xi=1}. \quad (1)$$

One of the results of the previous lectures was that H_{XXZ} commutes with $T(\xi)$, meaning that they share the same eigenvectors. Knowing the eigenvalues of the XXZ-model is the same as knowing the eigenvalues of the 6-vertex model. In this way these models are equivalent. Therefore we expect that the quantum affine algebra of Sl_2 solves not only the XXZ-model but also the 6-vertex model. Both models share the same R. The table should thus be written with the following addition

Model	Algebra
XXZ/6-Vertex	Quantum Affine Sl_2

The R-matrix computed in the final section will thus be the same for the XXZ-model and the 6-vertex model.

2 Constructing the Quantum Affine Algebra

Two ways of constructing a quantum affine algebra will be given in these lecture notes. One way consists of using Cartan matrices as building blocks, and their graphical representation in the form of Dynkin diagrams. We will see that Dynkin diagrams are a way of classifying Cartan matrices, but also a way of classifying root systems. (The other way to construct a quantum affine algebra is by means of a loop algebra. An affine Lie algebra can always be constructed as a central extension of the loop algebra of the corresponding simple Lie algebra. Both of these procedures will be outlined below.)

2.1 Dynkin Diagrams

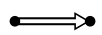
Any Cartan matrix satisfying the following relations gives rise to a finite-dimensional semi simple Lie algebra.

1. $C_{ii} = 2$.
2. $C_{ij} \leq 0$, $i \neq j$.
3. $\det(C) > 0$.
4. $\det(M(C) > 0)$, $M(C)$ is the principal minor.
5. C is diagonalizable: $C = BD$, where B is symmetric, D is diagonal.
6. $C_{ij} \in \mathbb{Z}$.

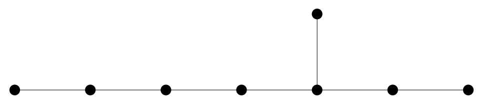
A way of classifying and visualizing different Lie algebras is by means of a Dynkin diagram. The rules for drawing a Dynkin diagram are as follows.

- For every row of the Cartan matrix, draw a node.
- Draw maximal (C_{ij}, C_{ji}) lines between the i^{th} and j^{th} node.
- When double or triple lines appear, an arrow can be drawn. If $|C_{ij}| > |C_{ji}|$ then the arrow points from node i to node j.

Another way of stating these rules is the following table. Here for each principle minor the corresponding Dynkin diagram is drawn.

$i - j$ **minor**	**Dynkin diagram**
$C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	
$C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$	
$C = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$	
$C = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$	

An example is given below. Deleting any node from the Dynkin diagram gives rise to another Lie algebra, which is just a sub algebra of the original one. It is important to note that the permutations don't matter in the diagram. Deleting a node from E8 will give E7. In the example we consider the Dynkin diagram of E8.

Exceptional Lie algebra E8	**Corresponding Dynkin diagram**
$C = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	

The following table gives the finite-dimensional simple Lie algebras of A_n, where n is the dimension of the Cartan matrix. The reason for only writing A_n here, is that we only use these algebras later on. There are of course Dynkin diagrams for all the Lie algebras, from A to E_8.

A_n, $n \geq 1$ 

Figure 1: Dynkin diagram of finite Cartan matrix A_n

2.2 Root Systems

Dynkin diagrams are actually the classification scheme of root systems. Since the definition of a root system already has been given, only a few examples of interest will be presented here. Namely, the root system of Sl_2 and Sl_3.

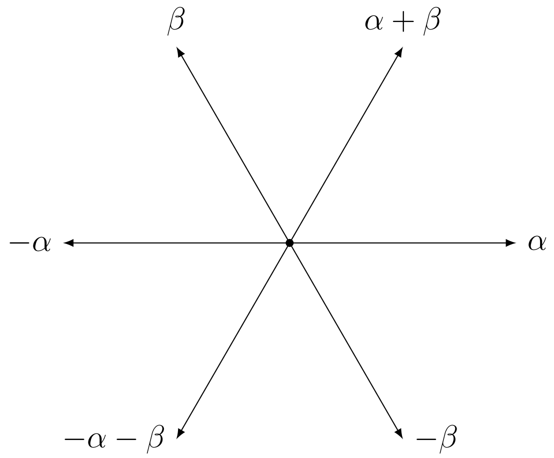


Figure 2: Root system of $Sl_3 = A_2$

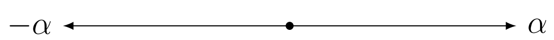


Figure 3: Root system of $Sl_2 = A_1$

There is a relation between the roots of a root system and the corresponding Cartan matrix. The relation is:

$$C_{ij} = \frac{2}{<\alpha_i, \alpha_i>} <\alpha_i, \alpha_j> .$$

Where $< .,. >$ is the standard in-product and α_i are the roots. We have established a way of building and classifying finite-dimensional simple Lie algebra using Cartan matrices, root systems and Dynkin diagrams. Now we will try to construct an affine Lie algebra, which means constructing an infinite-dimensional Lie algebra. We will proceed as follows: first we will change the conditions on the Cartan matrices and then we will see, using root systems, why our new matrices represent an infinite-dimensional algebra.

If we would drop conditions 3 and 4 of the Cartan matrices, the matrices we would get are called generalized Kac-Moody algebras. However, if we keep condition 4 and change condition 3 in the following way: $\det(C) > 0 \to \det(C) = 0$, we have an infinite-dimensional Lie algebra, also called affine. The Cartan matrix is still finite, it is the algebra that is infinite-dimensional.
Why is the algebra infinite-dimensional? This is best seen in a picture. Because $\det(C) = 0$, it is possible to have a so called null-root. Before, in the finite-dimensional case, it wasn't possible to find a vector a^i other then the null-vector satisfying

$$\sum a^i C_{ij} = 0.$$

However by putting the determinant zero, it is possible to find such a vector, other then the null-vector. This vector is called the null-root.

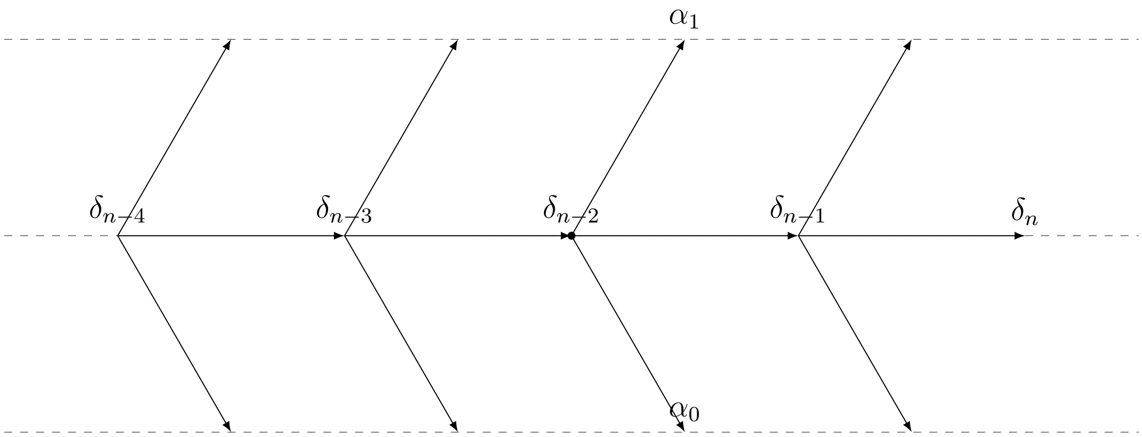
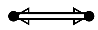
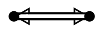
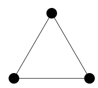
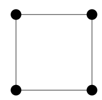
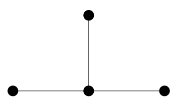
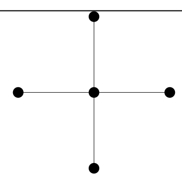


Figure 4: Root system of $\tilde{S}l_2 = \tilde{A}_1^1$

For every root $\alpha \in V$, where V is the root space, the element $\alpha + \delta \in V$ (this only holds for untwisted Lie algebras). The null root can be added an arbitrary number of times and this implies an infinite number of roots. Because we now have an infinite number of roots, it means that we have constructed an infinite-dimensional affine Lie algebra. The notation is as follows: $\tilde{S}l_2$, is the affinized algebra of Sl_2. Later on we want to use $\tilde{S}l_2$, for this reason we will give the structure here.

$\tilde{S}l_2 = A_1^1$	**Determinant**	**Principal minor**	**Dynkin diagram**
$C = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$	0	$C = 2$, $A_1 \leq A_1^1$	

The table below shows a few examples of a normal Lie algebra compared to its affinized version in Dynkin diagrams. In principle the reader should be able to draw the Dynking diagram and its corresponding affine version of any Lie algebra or given Cartan matrix. In the table only a few groups of A_n are treated.

Normal Lie algebra	**Untwisted Affine Lie algebra**
A_1	A_1^1 
A_2	A_2^1 
A_3	A_3^1 
A_4 	A_4^1 

2.3 Loop Algebra

The discussion about loop algebras will be short, since the route chosen in these lecture notes is by means of Dynkin diagrams. However, it is good to at least know that an affine Lie algebra always can be constructed as a central extension of the loop algebra of the corresponding simple Lie algebra. Affine Lie algebras can be quite interesting in CFT and String Theory, by the way they are constructed, namely using loop algebras. We start from a simple Lie algebra and consider the loop algebra L_g (a g valued function on a circle). Then the affine algebra is obtained by adding an extra dimension to the loop algebra and modifying a commutator in a non-trivial way. This amounts to the central extension.

Given a Lie algebra $\mathcal{G}_l$ with an $l \times l$ Cartan matrix, we define the loop algebra

$$L(g) = g \otimes \mathbb{C}[t, t^{-1}]$$

with $\mathbb{C}[t, t^{-1}]$ the algebra of the Laurent polynomials in t and t^{-1}. This uses the Lie bracket

$$[x \otimes P, y \otimes Q] = [x, y] \otimes PQ$$

where $P, Q \in \mathbb{C}[t, t^{-1}]$. We also induce: $(x \otimes P, y \otimes Q) = (x, y)PQ$, and define the two-cocycle on $L(\mathcal{G}_l)$

$$\psi(a, b) = Res(\frac{da}{dt}, b), a, b \in L(\mathcal{G}_l)$$

The residue of a Laurent series is $\text{Res}(\sum_{i \in \mathbb{Z}} a_i t^i) = a_{-1}$. Moreover, the two cocycle satisfies

$$\psi(a, b) = -\psi(a, b)$$
$$\psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0$$

Given the two-cocycle ψ, we may define the affine version $\tilde{L}(\mathcal{G}_l)$ as the central extension of $L(\mathcal{G}_l)$ by ψ. With a central extension the following is meant. Given a Lie algebra L, a central extension of L is a lie algebra $\tilde{L}$ such that

$$0 \to \mathfrak{a} \to \tilde{L} \to L \to 0, \mathfrak{a} \in Z(\tilde{L}).$$

For example, the Heisenberg algebra $\mathcal{H}$ is the algebra generated by p, q, z such that $[p, q] = z$, $[z, p] = 0 = [z, q]$. z is a central element of $\mathcal{H}$ and $\mathcal{H}$ is a central extension of $\mathbb{C}^2$

$$0 \to \mathbb{C}_z \to \mathcal{H} \to \mathbb{C}^2 \to 0.$$

In formulae, the statement that $\tilde{L}(\mathcal{G}_l)$ is the central extension of $L(\mathcal{G}_l)$ by ψ is the following: $\tilde{L}(\mathcal{G}_l) = L(\mathcal{G}_l) \oplus \mathbb{C}c$, where c is the central element, $[c, L(\mathcal{G}_l)] = 0$. Now the new commutator bracket has been changed in a non-trivial way such that it is now, in terms of the old one $[.,.] + \psi(.,.)c$. The last thing we need to do is add to our algebra the derivation operator d, which acts on $L(\mathcal{G}_l)$ as $t\frac{t}{dt}$.

2.4 Quantum Affine Algebra

We are ready to give the defining relations for the quantum affine algebra: $\tilde{U}_q(\mathcal{G}_l)$, where l is the rank of the Cartan matrix. These relations are for a general Lie algebra. We define $\tilde{U}_q(\mathcal{G}_l)$ to have $3(l+1)$ generators e_i, f_i, k_i, $i = 0, 1, ..., l$. Therefore, $Sl_2 = A_1^1$ has $l = 1$, 6 generators.

$$k_i k_j = k_j k_i$$
$$k_i e_j = q_i^{a_{ij}} e_j k_i$$
$$k_i f_j = q_i^{-a_{ij}} f_j k_i$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_j^{-1}}{q_i - q_i^{-1}}$$

(+ Serre relations)

Also, q_i corresponds to a single, a double or a triple line in the Dynkin diagram and gives q^1, q^2, q^3 etc. D_i corresponds to the diagonlized Cartan matrix. The a_{ij} correspond to the elements of the affine Cartan matrix. This is the reason we went trough the whole construction of the affine algebras using Cartan matrices.

The additional Serre relations are not mentioned here, since our discussion aims at the quantum affine Sl_2 algebra, and for this algebra the Serre relations don't give any extra information. In previous lectures we saw these relations in a more specific form. Also, in those relations a so called, derivation operator was always implicitly present. This derivation operator changes when going to the quantum case.

$$[d, e_i] = \delta_{i,0} e_i$$
$$[d, f_i] = -\delta_{i,0}$$
$$[d, k_i] = 0$$

And the derivation operator has the following co-multiplication

$$\Delta d = d \otimes \mathbb{I} + \mathbb{I} \otimes d$$

3 Universal R-matrix

This section is merely descriptive, just to give an impression of what the general case looks like. Drinfeld found the Universal R-matrix for $U_q \mathcal{G}_l$ using the quantum double construction. I will state the ingredients, the result, and comment on the different terms appearing in the formula.

First we introduce the universal R, depending on a formal affine parameter t. We then consider the following automorphism

$$D_t : \tilde{U}_q(\mathcal{G}_l) \otimes \mathbb{C}[t, t^{-1}] \to \tilde{U}_q(\mathbb{G}_l) \otimes \mathcal{C}[t, t^{-1}]$$
$$D_t(e_i) = t^{\delta_{i,0}} e_i$$
$$D_t(f_i) = t^{-\delta_{i,0}} f_i$$
$$D_t(k_i) = t^{\delta_{i,0}} k_i$$
$$D_t(d) = d$$

and new co-products

$$\Delta_t(a) = (D_t \otimes \mathbb{I})\Delta(a)$$
$$\Delta_t^{op}(a) = (D_t \otimes \mathbb{I})\Delta^{op}(a)$$

Then we define the following, where R is the universal R-matrix of $U_q\mathcal{G}_l$.

$$R(t) = (D_t \otimes \mathbb{I})R$$

Using these ingredients (the dual mapping has been left out, it can be found in [GRS] if you are interested), Drinfeld was able to construct the universal R-matrix.

$$R(t) = \exp[h(\sum_{i,j}^{l})(B_{ij})^{-1}H_i \otimes H_j] \times [\mathbb{I} + \sum_{i=1}^{l} 2\sinh hD_i e_i \otimes f_i + 2t \sinh h e_0 \otimes f_0 + ...]$$

- It is an expansion in the formal parameter t.
- H is a generator: $H_i = \frac{1}{2hD_i}\log(k_i^2)$.
- The small h appears in the quantum double construction when a dual map is used: $H \mapsto H^* = Hh$, $h \in \mathbb{C}$.
- $B = DA$, where A is the Cartan matrix, B is the symmetrized Cartan matrix and D is the diagonalized Cartan matrix.
- It satisfies the Yang-Baxter equation.
- It satisfies: $R(t)\rho^V \otimes \rho^W \Delta(a) = \rho^V \otimes \rho^W \Delta^{op}(a)R(t)$, here ρ is the finite-dimensional irreducible representation. (The fact that R satisfies this equation will be used in the explicit computation for $\tilde{U}_q Sl_2$)

4 Explicit R-matrix Computation for $\tilde{U}_q Sl_2$

We will go along the same lines as before. We start with the finite-dimensional Lie algebra of Sl_2, subsequently affinizing it and making it a quantum algebra. Recall our defining relations, and fill in $l = 2$, to obtain the relations for Sl_2. Let's be more specific. The Sl_2 algebra has three Chevalley generators E, F and H.

$$[E, F] = H$$
$$[H, E] = 2E$$
$$[H, F] = -2F$$

Spin generators: $E = S^+$, $F = S^-$ and $H = 2S^z$. The affine extension of this has six Chevalley generators.

$$E_0 = e^u F, E_1 = e^u F$$
$$F_0 = e^{-u} E, F_1 = e^{-u} F$$
$$H_0 = -H, H_1 = H$$

Here we call $x = \exp^{-u}$ the affinization parameter. Now we consider these generators in the Spin- $\frac{1}{2}$ representation. In this representation the generators will have the following form

$$E_0 = \begin{bmatrix} 0 & 0 \\ e^u & 0 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & e^u \\ 0 & 0 \end{bmatrix}$$
$$F_0 = \begin{bmatrix} 0 & e^{-u} \\ 0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & 0 \\ e^{-u} & 0 \end{bmatrix}$$
$$H_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, H_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then turn to the quantum deformation of the algebra and define: $K = q^H$. We can now use our previously obtained results for the defining relations of a general quantum affine algebra, and fill in the specific Cartan matrix. This gives the following relations

$$KE = q^2 EK$$
$$KF = q^{-2} FK$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

In the Spin-$\frac{1}{2}$ this only changes the last two matrices

$$K_0 = \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}, K_1 = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

We expect that the irreducible representation $(e^u, \frac{1}{2})$ of $\tilde{U}_q Sl_2$ should be intimately related to the Spin-$\frac{1}{2}$ representation of the Yang-Baxter generators, $A(u)$, $B(u)$, $C(u)$ and $D(u)$ of A^{6v}. This is indeed the case when we consider the following form:

$$A(u) = \frac{1}{2}(e^u \sqrt{qK} - e^{-u} \frac{1}{\sqrt{qK}})$$

$$B(u) = \frac{q - q^{-1}}{2\sqrt{q}} F\sqrt{K}$$

$$C(u) = \frac{q - q^{-1}}{2\sqrt{q}} E\sqrt{K}$$

$$D(u) = \frac{1}{2}(e^u \sqrt{q}\frac{1}{\sqrt{K}} - e^{-u}\frac{1}{\sqrt{q}}\sqrt{K})$$

Using these and our defining quantum affine algebra relations, we can find the relations for the Yang-Baxter algebra. It is left to the reader to check this. The quantum affine algebra $\tilde{U}_q Sl_2$ also has a bialgebra structure determined by the co-product.

$$\Delta(E_i) = E_i \otimes K_i + \mathbb{I} \otimes E_i$$

$$\Delta(F_i) = F_i \otimes \mathbb{I} + K_i^{-1} \otimes F_i$$

$$\Delta(K_i) = K_i \otimes K_i$$

Where $i = 0, 1$. We also need a transposed co-multiplication.

$$\Delta^{op}(E_i) = E_i \otimes \mathbb{I} + K_i \otimes E_i$$

$$\Delta^{op}(F_i) = F_i \otimes K_i^{-1} + \mathbb{I} \otimes F_i$$

$$\Delta^{op}(K_i) = K_i \otimes K_i$$

We can look for an intertwined R-matrix for the tensor product of two Spin-$\frac{1}{2}$ irreducible representations $(e^{u_1}, \frac{1}{2}) \otimes (e^{u_2}, \frac{1}{2})$ of $U_q Sl_2$. This should satisfy, just as we have seen in the general case for the universal R-matrix, the following relation

$$R(e^{u_1}, e^{u_2})\Delta_{e^{u_1}, e^{u_2}}(g) = \Delta^{op}_{e^{u_1}, e^{u_2}}(g) R(e^{u_1}, e^{u_2}),$$

where, $g \in \tilde{U}_q Sl_2$. Since we have the explicit matrix form for our generators we can compute the R-matrix explicitly. In class I tried to sandwich the R between states to get the elements of the R-matrix. This however can be done a lot faster, and is even trivial if you just notice that, when plugging in our explicit matrices into the equation, the equation becomes a linear equation. The solution can then be read easily. It is left as an exercise to the reader to plug the matrices into the equation. The result is the following

$$R^{01}_{01}(e^{u_1}, e^{u_2}) = e^{u_1 - u_2} - e^{u_2 - u_1},$$

$$R^{00}_{00}(e^{u_1}, e^{u_2}) = e^{u_1 - u_2} - q^{-1} e^{u_2 - u_1},$$

$$R^{10}_{01}(e^{u_1}, e^{u_2}) = q - q^{-1}.$$

Identifying $q = e^{i\gamma}$ and $e^u = e^{u_1 - u_2}$ we find

$$
\begin{aligned}
R_{01}^{01} &= e^u - e^{-u} = c \sinh u = a(u) \\
R_{00}^{00} &= e^{i\gamma} e^u - e^{-i\gamma} e^{-u} = c \sinh u + i\gamma = b(u) \\
R_{01}^{10} &= e^{i\gamma} - e^{-i\gamma} = c \sin u = c(u)
\end{aligned}
$$

where c is a constant. With $a(u)$, $b(u)$ and $c(u)$ we can construct exactly the R-matrix of the 6-vertex model!
It can also be shown using these explicit forms of the generators that the generators in the co-multiplication representation commute with the Hamiltonian of the XXZ-model.

$$[\Delta(\mathfrak{a}), H_{XXZ}] = 0, \mathfrak{a} \in \tilde{U}_q Sl_2$$

5 Conclusion

We started from the hint a few lectures ago that the underlying algebra of the 6-vertex model and the XXZ-model was the quantum affine algebra of Sl_2. Now we have turned this story around and first constructed a general affine algebra using Cartan matrices, root systems and their corresponding Dynkin diagrams. Then we took the general affine algebra and made it quantum, by giving the defining relations. In these relations we saw our previously used Cartan matrices reappearing. With the use of the general quantum affine algebra, Drinfeld was able to construct the Universal R-matrix. Since the quantum double construction is not part of these lectures, the construction has been very schematical, just to let the reader see the result. With the Universal R-matrix it should be possible in theory to go immediately to the desired case, namely the R-matrix for $\tilde{U}_q Sl_2$. However, in practice it is easier to use the construction in the previous section. Eventually we did find the R-matrix of the 6-vertex model, and thus the R-matrix of the XXZ-model, using $\tilde{U}_q Sl_2$ and its relations between the generators. Therefor it has been shown that the $\tilde{U}_q Sl_2$ algebra solves both models.

6 Literature

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