Recap Lecture 3: Coordinate Bethe Ansatz XXX-model

Hamiltonian:

$$\hat{H} = -J \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} = -J \sum_{n=1}^{L} \left(\frac{1}{2}(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) + S_n^z S_{n+1}^z \right)$$

Symmetry group:

$$G = \mathbb{Z}/L\mathbb{Z} \times SU(2)$$

Using only $U_z(1)$:

$$S_{tot}^z = \frac{L}{2} - M \text{ and } \mathcal{H} = \bigoplus_{M=0}^{L} \mathcal{H}_M$$

Solve model:

1) M = 1

$$|\psi_k\rangle = \sum_n f(n)\,|n\rangle \text{ with } f(n) = e^{ikn}$$

$$E - E_0 = J(1 - \cos k) \Rightarrow \text{ degenerate states: } |0\rangle \text{ and } |\psi(k=0)\rangle$$

2) M = 2

$$|\psi_{k_1,k_2}\rangle = \sum_{n_2>n_1} f(n_1,n_2)\,|n_1,n_2\rangle\,,$$

where

$$f(n_1,n_2) = Ae^{i(k_1n_1+k_2n_2)} + Be^{i(k_1n_2+k_2n_1)}.$$

Energy:

$$E = J\sum_{i=1}^{M}(1-\cos k_i) + E_0 \tag{1}$$

Demand Ansatz is eigenstate:

$$e^{i\theta} := \frac{A}{B} = -\left(\frac{e^{i(k_1+k_2)}+1-2e^{ik_1}}{e^{i(k_1+k_2)}+1-2e^{ik_2}}\right) \tag{2}$$

Demand periodicity: $f(n_1,n_2) = f(n_2,n_1+L)$

$$\begin{cases} k_1L = \theta + 2\pi m_1,\ m_1 \in \{0,\dots,L-1\} \\ k_2L = -\theta + 2\pi m_2,\ m_2 \in \{0,\dots,L-1\} \end{cases} \tag{3}$$

Solution, use equations (1) - (3) to:
1) Find "L choose M" different eigenstates, characterized by $m \in (\mathbb{Z}/L\mathbb{Z})^M$.
2) For every m, find a different solution $\vec{k} \in \mathbb{C}^M$.