More About the Yang Baxter Equation

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Abstract

In this notes we first review the construction of the Yang Baxter equation in a pedagogical way. The monodromy matrix is introduced, along with the proof of the RTT=TTR relation. From these results we introduce some simple notions of integrability and the connection between the 6-vertex model and the XXZ model. Then we review the construction of a non-trivial faithful representation of the braid group by means of the Boltzmann weights R and the Yang Baxter equation. Finally we briefly introduce the Yang Baxter algebra, it's co-product and it's adjoint representation.

1 Notation

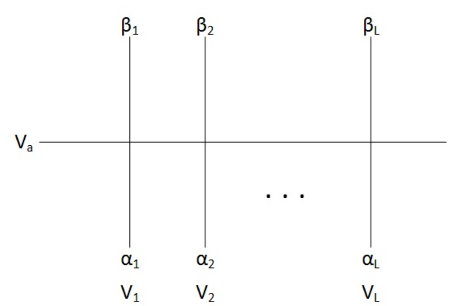
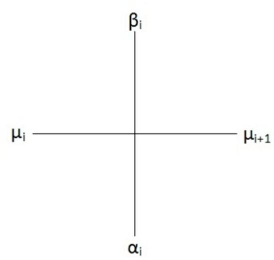


Figure 1: Row of the 6-vertex model lattice, with auxiliary space V_a and vertical spaces V_1,.. V_L

First we introduce the notation that will be used through out the presentation, which will be mostly based on [1] and [2]. As was previously seen, we divided the lattice of the six vertex model into rows (recall Figure 1). The vector spaces corresponding to the vertical states will be denoted by V_1,..., V_L. The tensor product of these L vertical spaces will be called the "quantum space" and will be denoted as $\mathrm{H}^{(L)} = V_1 \otimes V_2 \otimes ... \otimes V_L$, while the horizontal space V_a will be called the "auxiliary space".

As we have seen in previous presentations, a Boltzmann weight R will be associated to each vertex of the lattice. In the notation used in this text the weight of the i-th vertex will be represented by using the notation illustrated in Figure 2.



$$= R^{\mu_{i+1}\beta i}_{\mu_i \alpha_i} =_i \langle \beta_i | \otimes_a \langle \mu_{i+1} | \, R_{ai} | \mu_i \rangle_a \otimes | \alpha_i \rangle_i$$

Figure 2: Notation for the Boltzmann weight of the i-th vertex

When R is accompanied with 2 subscripts, for example in R_{ai}, the subscripts a and i are indicating the vector spaces in which R is acting on; in this case, the auxiliary space V_a and the vertical space V_i. In this way, we should think of R_{ai} as an operator acting in the following way:

$$R_{ai} : V_a \otimes V_i \to V_a \otimes V_i$$

When R has 4 indexes, for instance in $R^{\mu_{i+1}\beta i}_{\mu_i \alpha_i}$, these indexes are labels for the basis vectors of the spaces R is acting on. In the case of the six vertex model we know that these indexes can take only two values: + or -.

Previously we defined the transfer matrix t as setting up the evolution in between rows; by acting on the lower vertical states $|\alpha\rangle$, t produces the vertical states $|\beta\rangle$ (see Figure 1). In the index notation the transfer matrix will be given by:

$$\langle \beta | \, t \, | \alpha \rangle = \sum_{\mu' s} R^{\mu_1 \ \beta_L}_{\mu_L \ \alpha_L} R^{\mu_L \ \beta_{L-1}}_{\mu_{L-1} \ \alpha_{L-1}} \cdots R^{\mu_3 \ \beta_2}_{\mu_2 \ \alpha_2} R^{\mu_2 \ \beta_1}_{\mu_1 \ \alpha_1} \tag{1}$$

where we see that we are summing over all possible horizontal states, and where the periodic boundary condition (namely $\mu_{L+1} = \mu_1$) is also taken into account.

Using the subscript notation for R this expression can be written as:

$$t = \mathrm{tr}_a \left[R_{a\ L} R_{a\ L-1} \cdots R_{a\ 2} R_{a\ 1} \right] \tag{2}$$

where we see that the periodic boundary condition is imposed by means of the trace over the auxiliary space.

2 The Yang Baxter Equation

Once the transfer matrix was introduced, we proceeded to study under what circumstances 2 different transfer matrices commute. For this we will consider transfer matrices t and t', with auxiliary spaces V_a and V_b, and with corresponding Boltzmann weights R and R' (in principle R and R' are not necessarily the same).

Using the definition of the transfer matrix we see that tt' and $t't$ can be written in the following way:

$$t\,t' = \mathrm{tr}_{a\times b}\,[R_{a\ L} R'_{b\ L} \cdots R_{a\ 1} R'_{b\ 1}] \tag{3}$$

$$t'\,t = \mathrm{tr}_{a\times b}\,[R'_{b\ L} R_{a\ L} \cdots R'_{b\ 1} R_{a\ 1}] \tag{4}$$

where we used the fact that Boltzmann weights with different auxiliary spaces, acting on different vertical spaces, commute.

In order to get a better grasp of what is happening, we will represent these equations graphically as seen in Figure 3.

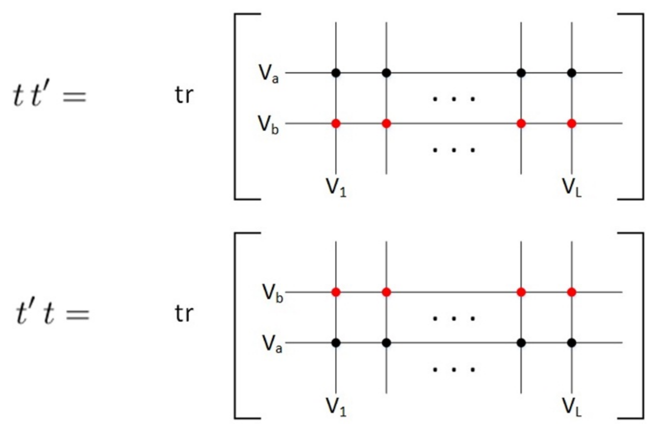


Figure 3: Graphical representation of Equations (3) and (4), the Boltzmann weights of t and t' are represented by black and red dots respectively

Equations (3) and (4) are equal if and only if there exist an invertible matrix $M_{a\ b}$, such that:

$$R'_{b\ i} R_{a\ i} = M_{a\ b} R_{a\ i} R'_{b\ i} {M_{a\ b}}^{-1} \qquad \forall i = 1, ..., L \tag{5}$$

It is easy to check that by substituting this equation into (4) we get (3).

In order to represent graphically what this equation is doing, we will first introduce a graphical notation for $M_{a\,b}$ and it's inverse (see Figure 4).

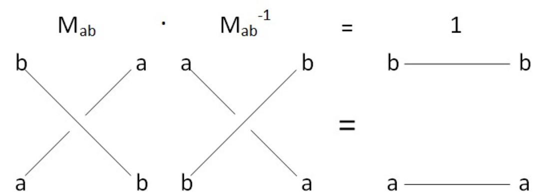


Figure 4: Graphical notation for $M_{a\,b}$ and it's inverse. We see that in this notation $M_{a\,b}M^{-1}_{a\,b} = \mathbb{1}$

From now on we will assume that the matrix $M_{a\,b}$ has the same structure as a Boltzmann weight acting on $V_a \otimes V_b$, therefore we will now write it as $M_{a\,b} = R'''_{a\,b}$. Acting with $R'''_{a\,b}$ on the right at both sides of equation (5), we finally get:

$$R'_{b\,i} R_{a\,i} R''_{a\,b} = R''_{a\,b} R_{a\,i} R'_{b\,i} \tag{6}$$

This equation is known as the Yang Baxter equation. The graphical version of this expression is shown in Figure 5, where we see that moving $R'''_{a\,b}$ from one side of the i-th vertical line to the other, is always accompanied by a switch in the position of the Boltzmann weights of the vertical space V_i.

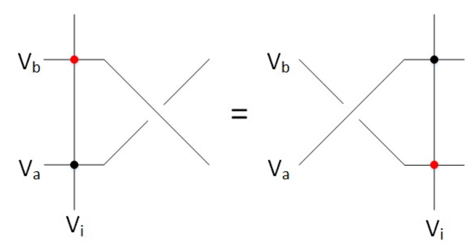


Figure 5: Graphical representation of the Yang Baxter equation

By doing the following changes in notation:

$$\begin{array}{ccc} R''_{a\,b} \rightarrow R_{12} & R_{a\,i} \rightarrow R'_{13} & R'_{b\,i} \rightarrow R''_{23} \\ V_a \rightarrow V_1 & V_b \rightarrow V_2 & V_i \rightarrow V_3 \end{array}$$

we get the following expression, which is the one most often used in the literature:

$$R_{12}R'_{13}R''_{23} = R''_{23}R'_{13}R_{12} \tag{7}$$

To interpret this equation we can think of R_{ij} with $i, j \in \{1, 2, 3\}$ and $i \neq j$, as an operator acting on $V_1 \otimes V_2 \otimes V_3$; it will act as a Boltzmann weight on the spaces V_i and V_j , and as the identity on the remaining vector space.

In index notation, the equation above is written as follows:

$$\sum_{j_1,j_2,j_3} R^{k_1k_2}_{j_1j_2} R'^{j_1k_3}_{i_1j_3} R''^{j_2j_3}_{i_2i_3} = \sum_{j_1,j_2,j_3} R''^{k_2k_3}_{j_2j_3} R'^{k_1j_3}_{j_1i_3} R^{j_1j_2}_{i_1i_2}$$

So far we have been careful not to assume any particular structure for the Boltzmann weights R. The reason for this is to illustrate to the reader that the Yang Baxter equation is not restricted exclusively to the 6-vertex model.

In particular, for the six vertex model the Boltzmann weight R can be written as a matrix depending on parameters a, b and c:

$$R^{(6v)} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

were we used the upper indexes of $R^{\mu_{i+1}\beta i}_{\mu_i\alpha_i}$ as labels for the columns and the lower indexes for the rows of the matrix. Therefore, equation (7) corresponds to a matrix equation.

Due to the fact that our Boltzmann weights R, R' and R'' have this same structure, we can label them in terms of their parameters:

$$R = R^{(6v)}(a, b, c) \qquad R' = R^{(6v)}(a', b', c') \qquad R'' = R^{(6v)}(a'', b'', c'')$$

As was seen in the previous week, the Yang Baxter equation in the case of the 6-vertex model will correspond to 64 equations that due to symmetries in the system get reduced to 3 :

$$ac'a'' = bc'b'' + ca'c''$$
$$ab'c'' = ba'c'' + cc'b''$$
$$cb'a'' = ca'b'' + bc'c''$$

From this system of equations we found that:

$$\triangle(a, b, c) = \triangle(a', b', c') = \triangle(a'', b'', c'') \tag{8}$$

where:

$$\triangle(a, b, c) = \frac{a^2 + b^2 - c^2}{2ab} \tag{9}$$

Therefore $\triangle$ has to be the same for the 3 matrices, regardless of the particular choices of a's, b's and c's.

By introducing a parametrization for the different a's, b's and c's, that automatically satisfies (9) we can describe R in terms of a scaling parameter λ:

$$\begin{aligned}
&a = \rho \sinh(\lambda + \phi) \\
&b = \rho \sinh(\lambda) \\
&c = \rho \sinh(\phi) \\
&\triangle = \cosh \phi
\end{aligned}$$

As was seen in the previous week, ρ corresponds to an overall scaling and thus it won't matter, and ϕ is fixed in order for $\triangle$ to be fixed. By doing this we can write each Boltzmann weight only in terms of its scaling parameter:

$$R = R(\lambda) \qquad R' = R(\lambda') \qquad R'' = R(\lambda'')$$

Replacing the parametrization above in the system of equations we found from the Yang Baxter equation, we find that one of the scaling parameters can be written in terms of the other 2.

By playing with the notation of the λ's, the Yang Baxter equation for the six vertex model, using this parametrization, can be written as:

$$R_{12}(\lambda) R_{13}(\lambda + \lambda') R_{23}(\lambda') = R_{23}(\lambda') R_{13}(\lambda + \lambda') R_{12}(\lambda) \tag{10}$$

3 The RTT=TTR Relation

We now introduce the concept of monodromy matrix, which we will denote by T, and is defined as:

$$T(\lambda) = R_{a\,L} R_{a\,L-1} \cdots R_{a\,2} R_{a\,1} \tag{11}$$

Due to it's definition, it is clear that the monodromy matrix is related to the transfer matrix in the following way:

$$t(\lambda) = \mathrm{tr}_a [T(\lambda)] \tag{12}$$

From the definition of the monodromy matrix we now proceed to prove what is called as the RTT=TTR relation.

For this we will abuse the notation a little bit by introducing an index to T, in order to make it explicit that T_a corresponds to the monodromy matrix with auxiliary space V_a, while T_b has auxiliary space V_b. From equation (11) we have that:

$$
\begin{aligned}
&R_{ab}\left(\lambda-\lambda'\right) T_a\left(\lambda\right) T_b\left(\lambda'\right)\\
&\qquad\qquad = R_{ab}\left(\lambda-\lambda'\right) R_{a\ L}\left(\lambda\right)\cdots R_{a\ 1}\left(\lambda\right) R_{b\ L}\left(\lambda'\right)\cdots R_{b\ 1}\left(\lambda'\right)\\
&\qquad\qquad = R_{ab}\left(\lambda-\lambda'\right) R_{a\ L}\left(\lambda\right) R_{b\ L}\left(\lambda'\right)\cdots R_{a\ 1}\left(\lambda\right) R_{b\ 1}\left(\lambda'\right)
\end{aligned}
$$

This will be the RTT side of the equation. In order to see graphically what is going on, we will use pictures once again. The graph that represents the equation above is shown in Figure 6 a).

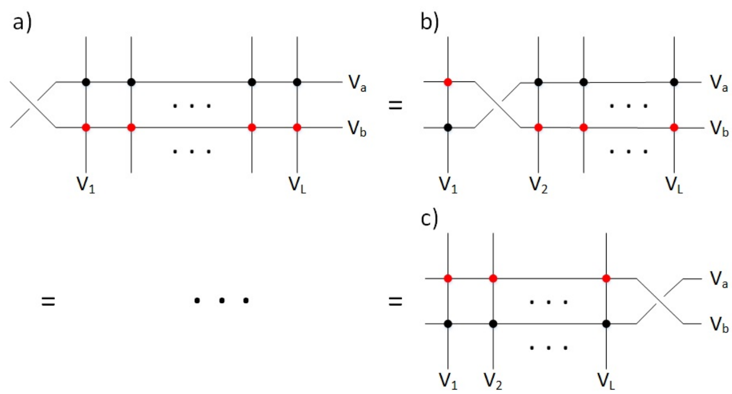


Figure 6: Graphical representation of a) the RTT side of the relation. b) result of using the Yang Baxter equation once. c) TTR side of the relation.

We see that we can apply the Yang Baxter equation once, in order to move R_{ab} to the right:

$$
\begin{aligned}
&R_{ab}\left(\lambda-\lambda'\right) T_a\left(\lambda\right) T_b\left(\lambda'\right)\\
&= R_{b\ L}\left(\lambda'\right) R_{a\ L}\left(\lambda\right) R_{ab}\left(\lambda-\lambda'\right) R_{a\ L-1}\left(\lambda\right) R_{b\ L-1}\left(\lambda'\right)\cdots R_{a\ 1}\left(\lambda\right) R_{b\ 1}\left(\lambda'\right)
\end{aligned}
$$

Graphically what we have done is to move R_{ab} one place to the right, therefore we have to switch the Boltzmann weights of the corresponding vertical line (recall Figure 5). The result of applying the Yang Baxter equation once is shown in Figure 6 b).

By applying the Yang Baxter equation several times, we can move R_{ab} to the right:

$$
R_{ab}\left(\lambda-\lambda'\right) T_a\left(\lambda\right) T_b\left(\lambda'\right) = R_{b\ L}\left(\lambda'\right)\cdots R_{b\ 1}\left(\lambda'\right) R_{a\ L}\left(\lambda\right)\cdots R_{a\ 1}\left(\lambda\right) R_{ab}\left(\lambda-\lambda'\right)
$$

Using (11) once again, we get the TTR side of the relation:

$$R_{ab}\left(\lambda-\lambda'\right)T_a\left(\lambda\right)T_b\left(\lambda'\right)=T_b\left(\lambda'\right)T_a\left(\lambda\right)R_{ab}\left(\lambda-\lambda'\right) \tag{13}$$

The TTR side of the equation is illustrated in Figure 6 c), where it is clear that all Boltzmann weights of V_a and V_b of a same vertical line have switched positions by using the Yang Baxter equation several times.

It is easy to see that Figure 5 and Figures 6 a) and c) are quite similar; the Yang Baxter equation can be seen as "local" since it only involves one vertical vector space V_i, while the RTT=TTR can be seen as a "global" relation since it is related to $\mathrm{H}^{(L)}=V_1\otimes V_2\otimes\ldots\otimes V_L$. These 2 equations are very important due to the fact that they are the basis for the quantum inverse scattering method.

Equation (13) can also be written in index notation in the following way:

$$\sum_{j_1,j_2} R\left(\lambda-\lambda'\right)^{k_1k_2}_{j_1j_2}T\left(\lambda\right)^{j_1}_{i_1}T\left(\lambda'\right)^{j_2}_{i_2}=\sum_{j_1,j_2}T\left(\lambda'\right)^{k_2}_{j_2}T\left(\lambda\right)^{k_1}_{j_1}R\left(\lambda-\lambda'\right)^{j_1j_2}_{i_1i_2}$$

From equation (13) we also have that:

$$T_a\left(\lambda\right)T_b\left(\lambda'\right)=R^{-1}{}_{ab}\left(\lambda-\lambda'\right)T_b\left(\lambda'\right)T_a\left(\lambda\right)R_{ab}\left(\lambda-\lambda'\right)$$

By applying $\mathrm{tr}_{a\times b}$ on both sides we get:

$$\left[t\left(\lambda\right),t\left(\lambda'\right)\right]=0 \qquad \forall\lambda,\lambda' \tag{14}$$

4 Integrability and the XXZ Model

The simplest notion of integrability comes from Liouville's theorem. This theorem states that if a system with a $2n$-dimensional phase space has n functions F_i such that they have vanishing Poisson brackets:

$$\left\{F_i,F_j\right\}_{P.B.}=0 \tag{15}$$

and the Hamiltonian is one of this functions F_i, then the system can be solved by quadratures [3].

This theorem is applicable for simple classical systems like the harmonic oscillator, etc. For more complicated systems, like the ones seen in this course, the generalization is not that direct. However, it introduces us to a property shared by quantum integrable systems: The existence of mutually Poisson commuting "higher Hamiltonians", like the ones shown in equation (15) [4].

From equation (14) we see that the commutation of transfer matrices in the six vertex model, can be seen as the corresponding vanishing Poisson brackets of higher Hamiltonians. In equation (15) each F_i will correspond to a conserved quantity, for the case of the six vertex model the equivalence can be seen by thinking of the transfer matrix as an "evolution operator" between the horizontal lines of the lattice, in this way any $t(\lambda)$ will determine a particular evolution with a corresponding conserved quantity.

Instead of threating $t(\lambda)$ as the generating functional for conserved charges, we will use its logarithm. Expanding around $\lambda = 0$ we have that:

$$\ln\ t(\lambda) = \sum_{n=0}^{\infty} J_n \lambda^n \tag{16}$$

Substituting (16) in (14) we get:

$$[J_n, J_m] = 0 \tag{17}$$

This expansion coefficients J_n should be interpreted as conserved densities [2]. This conserved densities then can be used to solve the system, provided we have enough of them. As a trivial check we see that from their definition, the coefficients J_n have no functional dependence on the spectral parameter λ, and therefore do not depend directly on the parametrization, which is what is expected for physical quantities.

As an introduction to the hand-in exercise of this week we will mention the results for the first 2 conserved densities since they are of considerable importance.

The first conserved density will be given by:

$$J_0 = \ln\ t(0)$$

Using the definition of the monodromy matrix, the R matrix of the six vertex model and the parametrization introduced above, it can be shown that the action of e^{J_0} on the vertical states $|\alpha\rangle = |\alpha_1, \cdots, \alpha_L\rangle$ is given by:

$$e^{J_0}\,|\alpha\rangle = t(0)\,|\alpha_1, \cdots, \alpha_L\rangle = {c_0}^L\,|\alpha_L, \alpha_1, \cdots, \alpha_{L-1}\rangle$$

From this we conclude that J_0 will be related to the momentum operator since the momentum operator is the generator of translations, and basically, the equation above corresponds to a translation of all the states one site to the right [1].

For the second conserved density the result is that:

$$J_1 = \left.\frac{d \ln t(\lambda)}{d\lambda}\right|_{\lambda=0} = \text{Const.} + \text{Const.} \sum_{i=1}^{L} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right] \tag{18}$$

Quite surprisingly the expression on the right is no other than the Hamiltonian of the XXZ Hamiltonian [1]. This implies that all higher conserved quantities constructed in this way will be conserved charges for the XXZ model.

The appearance of the XXZ Hamiltonian in this treatment of the 6 vertex model is not that new; in the previous week we saw that the transfer matrix of the 6-vertex model and the XXZ Hamiltonian commute with each other [1], and therefore it is not entirely unexpected to find the Hamiltonian in the expansion of the transfer matrix (recall that (14) implies that $t(\lambda)$ will commute with every term in its expansion).

Due to this link between the 2 models we have that the commutation of transfer matrices in the 6 vertex model, will correspond to commuting higher Hamiltonians J_n for the XXZ model. Moreover, knowing the eigenvalues of the transfer matrix will be equivalent to knowing the eigenvalues of the higher Hamiltonians J_n of the XXZ model.

This relation between the 2 models may appear at first as a coincidence. However, that is not the case. This is due to a general principle that states the equivalence between d dimensional classical lattice models (in our case it is the 2-dimensional six vertex model) and a $d-1$ dimensional quantum lattice model (in this case the XXZ spin chain) [4].

5 The Braid Group and the Yang Baxter Equation

We now proceed to make a connection between the braid group and the Yang Baxter equation. For this we will need to rewrite (10) in a different form.

Recall that we defined R as $R : V_1 \otimes V_2 \to V_1 \otimes V_2$, we will now introduce $\Re$ as being defined by:

$$\Re = PR \tag{19}$$

where P stands for the permutation operator. From it's definition it is clear that $\Re$ acts in the following way:

$$\Re : \quad V_1 \otimes V_2 \to \quad V_2 \otimes V_1 \tag{20}$$

$$e_i^{(1)} \otimes e_j^{(2)} \to e_j^{(2)} \otimes e_i^{(1)} \tag{21}$$

where $e_i^{(1)}$ and $e_j^{(2)}$ are basis vectors of V_1 and V_2 respectively.

Due to the definition of $\Re$ we also have that it's components will be related to the components of R as follows:

$$\Re_{ij}^{kl} = R_{ij}^{lk} \tag{22}$$

Using the definition of $\Re$, we can rewrite the Yang Baxter equation (10) in the following way:

$$(\mathbb{1} \otimes \Re(\lambda))(\Re(\lambda+\lambda') \otimes \mathbb{1})(\mathbb{1} \otimes \Re(\lambda')) \tag{23}$$
$$= (\Re(\lambda') \otimes \mathbb{1})(\mathbb{1} \otimes \Re(\lambda+\lambda'))(\Re(\lambda) \otimes \mathbb{1}) \tag{24}$$

On the other hand, the braid group is defined as the group generated by the elements σ_i with $i \in 1, ..., L-1$, satisfying the following properties:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{25}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| \geq 2 \tag{26}$$
$$\sigma_i {\sigma_i}^{-1} = {\sigma_i}^{-1} \sigma_i = \mathbb{1} \tag{27}$$

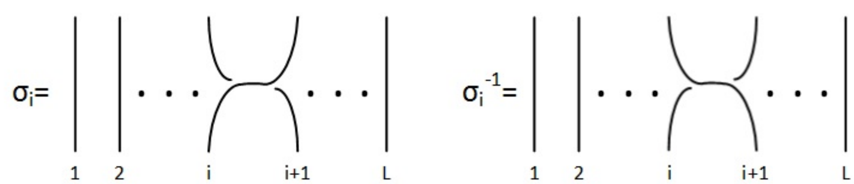


Figure 7: Graphical interpretation of σ_i and ${\sigma_i}^{-1}$

To understand what this means, we can interpret this graphically by imaging a set of L vertical lines. The identity element will correspond to leaving these L vertical lines intact, while we can think of each element of the group σ_i as being an over-crossing of the i-th line over the $i+1$-th line, while ${\sigma_i}^{-1}$ would correspond to an under-crossing (see Figure 7) .

We will now proceed to illustrate what equations (25), (26) and (27) mean. Equation (25) is also called the "Braid equation" and its graphically represented by Figure 8. The equivalence of both sides of the equation can be seen by comparing the location of the ends of each string, as well as both sides having the same structure if the lines were thought of as ropes being stretchered.

Equation (26) means that if i and j are separated enough it doesn't matter in which order we apply σ_i and σ_j. Meanwhile, equation (27) is graphically represented by Figure 9, where it is easy to see that $\sigma_i \sigma_i^{-1}$ and $\sigma_i^{-1} \sigma_i$ correspond to the identity; indeed we observe that by stretching the ropes and comparing the ends of each line we see that the 2 graphs on the left of Figure 9 are identical to the identity.

We will now introduce the following operator, which will allow us to make a connection between the braid group and the Yang-Baxter equation:

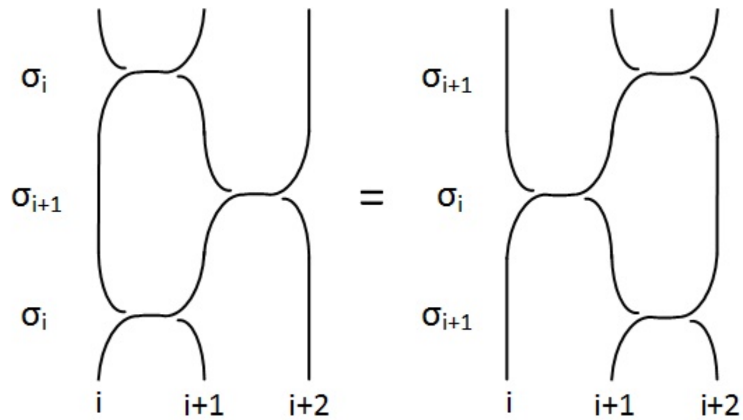


Figure 8: Graphical interpretation of the Braid equation (25)

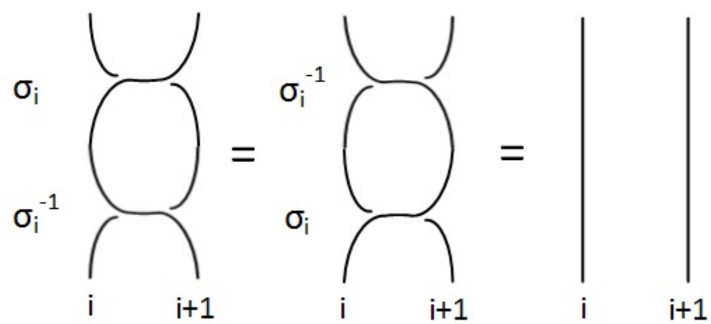


Figure 9: Graphical interpretation of $\sigma_i {\sigma_i}^{-1} = {\sigma_i}^{-1} \sigma_i = \mathbb{1}$

$$\Re_i\left(\lambda\right) = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \overbrace{\Re\left(\lambda\right)}^{(i,i+1)} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \tag{28}$$

As we can see in the equation above, $\Re_i$ will act as the identity except on V_i and V_{i+1}, where it acts as $\Re(\lambda)$.

Using this operator we can write equation (24) in a more general way:

$$\Re_{i+1}\left(\lambda\right)\Re_i\left(\lambda+\lambda'\right)\Re_{i+1}\left(\lambda'\right) = \Re_i\left(\lambda'\right)\Re_{i+1}\left(\lambda+\lambda'\right)\Re_i\left(\lambda\right) \tag{29}$$

Comparing this equation with the Braid equation (25) we see that they have a similar behavior, except for the fact that here the $\Re_i$ operator has an explicit dependence on the scaling parameter λ, which is something we do not have on the Braid equation.

We also see that the operator $\Re_i$ as defined in equation (28) satisfies that:

$$\Re_i(\lambda)\,\Re_j(\lambda') = \Re_j(\lambda')\,\Re_i(\lambda) \qquad |i-j| \geq 2 \tag{30}$$

This equation is equivalent to equation (26) of the Braid group.

From this two equations we see that we can construct a representation for the Braid group, in which the Braid equation is given by the Yang Baxter equation. However, in order to achieve this it is necessary to remove the dependence of equation(29) on the scaling parameter λ.

There are 2 solutions to this problem. The first one is to set $\lambda = \lambda' = 0$, while the second one is to take $\lambda = \lambda'$ with $|\lambda| = \infty$. For the first solution we have that $R(0) = P$, and therefore $\Re(0) = PP = \mathbb{1}$, this will correspond to the trivial representation since every element σ_i of the braid group is represented by $\Re_i(0) = \mathbb{1}$.

The second solution is much more interesting and is known as the "Braid limit". In order to illustrate how this comes about we will use the following parametrization:

$$\begin{aligned} a(\lambda) &= \sinh(\lambda + i\gamma) \qquad & b(\lambda) &= \sinh(\lambda) \\ c(\lambda) &= i\sin(\gamma) \qquad & \Delta &= \cos(\gamma) \end{aligned}$$

Using this parametrization one can show that:

$$\lim_{\lambda\to\pm\infty} e^{-|\lambda|}\Re(\lambda) \sim P\exp\left[\pm\frac{i\gamma}{2}\sigma^z\otimes\sigma^z\right]$$

This equation tells us that $\Re(\lambda)$ in the braid limit is proportional to P, therefore applying $\Re$ twice or applying $\Re$ followed by $\Re^{-1}$, will in both cases produce something proportional to the unity. This is not the behavior we want in a faithful representation of the braid group, this is because $\sigma_i\sigma_i^{-1} = \mathbb{1}$ but $\sigma_i\sigma_i \neq \mathbb{1}$ as can be seen in Figure 10.

We see that using the parametrization introduced above, the Boltzmann weights behave in the following way when taking $\lambda \to \infty$:

$$a(\lambda) \sim \frac{1}{2}e^{\lambda}e^{i\gamma} \qquad b(\lambda) \sim \frac{1}{2}e^{\lambda} \qquad c(\lambda) = i\sin(\gamma)$$

From this we see that a and b scale exponentially with the scaling parameter, while c is fixed. This is a bit unnatural since, as we have seen in previous lectures, the system is invariant under the overall scaling of all 3 variables.

In order to correct this behavior and with the aim to produce a faithful representation of the braid group, we now used the trick proposed by Jimbo

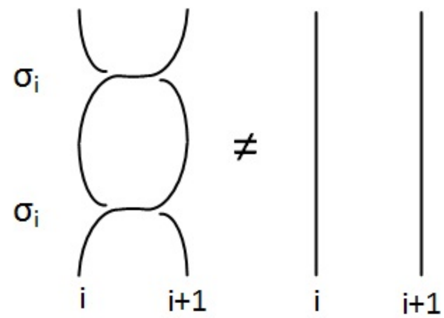


Figure 10: Graphical interpretation of $\sigma_i\sigma_i$, the result is clearly not equal to $\mathbb{1}$ due to the "knot" in the middle

[1][4], in which we rescale the basis vectors of the spaces we are acting on, namely:

$$\widetilde{e_r}(\lambda) = f_r(\lambda)\, e_r(\lambda) \qquad r = 1, \ldots, L \tag{31}$$

Recalling that $\Re$ is an operator that acts in the following way:

$$\Re\left(e_{r_1}^{(\lambda)} \otimes e_{r_2}^{(\lambda')}\right) = \Re_{r_1\, r_2}^{r_2{}' r_1{}'}(\lambda, \lambda')\, e_{r_2{}'}(\lambda') \otimes e_{r_1{}'}(\lambda) \tag{32}$$

we see that for each vector $e_{r'}(\lambda)$, we have that $\widetilde{\Re}(\lambda)$ has to be scaled by a factor of $1/f_r'(\lambda)$.

Taking this into account we have that under the rescaling $\Re$ transforms into:

$$\widetilde{\Re}_{r_1\, r_2}^{r_2{}' r_1{}'}(\lambda, \lambda') = \frac{f_{r_1}(\lambda)\, f_{r_2}(\lambda')}{f_{r_1{}'}(\lambda)\, f_{r_2{}'}(\lambda')} \Re_{r_1\, r_2}^{r_2{}' r_1{}'}(\lambda - \lambda') \tag{33}$$

where the factors of the type $f_r(\lambda)$ found on the numerator correspond to the rescaling of basis vectors $e^r(\lambda)$, which rescale as $\widetilde{e^r}(\lambda) = e^r / f_r(\lambda)$ in order to preserve the normalization of the product between basis vectors.

Previously we have seen that the Yang Baxter equation has in one if its $\Re$'s a dependence on the addition/subtraction of scaling parameters λ and λ'. In general, we would like this property to be preserved under the rescaling of $\Re$. To do this it is essential to make an adequate choice of the functions $f_r(\lambda)$, in this case we will use:

$$f_r(\lambda) = e^{\alpha\lambda r}$$

Another property that we would like $\widetilde{\Re}$ to preserve is the conservation law at each vertex. This is equivalent to imposing the following condition:

$$\widetilde{\Re}_{r_1\, r_2}^{r_2{}' r_1{}'}(\lambda, \lambda') = 0 \text{ unless } \; r_1 + r_2 = r_1{}' + r_2{}'$$

Taking these 2 considerations into account, we find that $\Re$ is rescaled in the following way:

$$\widetilde{\Re}^{r_2{}' r_1{}'}_{r_1\, r_2}\left(\lambda-\lambda', \alpha\right) = e^{\alpha\left(\lambda-\lambda'\right)\left(r_1 - r_1{}'\right)} \Re^{r_2{}' r_1{}'}_{r_1\, r_2}\left(\lambda-\lambda'\right) \tag{34}$$

this expression satisfies the Yang Baxter equation for any α. By convention we will choose the following normalization for $\Re$, which is the one will use for following presentations in this topic:

$$\Re \equiv 2e^{-i\gamma/2} \lim_{\lambda\to\infty} \; e^{-\lambda}\widetilde{\Re}\left(\lambda, \alpha=1\right) \tag{35}$$

Using the R matrix of the 6 vertex model and replacing (34) into (35) we find the following expression for $\Re$ in the braid limit:

$$\Re = \begin{bmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & q^{-1/2} & q^{-1/2}\left(q-q^{-1}\right) & 0 \\ 0 & 0 & 0 & q^{1/2} \end{bmatrix} \tag{36}$$

where $q = e^{i\gamma}$.

This newly defined $\Re$ for the braid limit satisfies the property that for $\gamma \neq 0$, $\Re^2 \neq \mathbb{1}$ and therefore it will create a faithful representation of the braid group. It is also interesting to check that for $\gamma = 0$ we have that $\Re = P$, but since $\Re = PR$, this will correspond once again to the trivial case.

The inverse of $\Re$ will be defined as:

$$\Re^{-1} \equiv -2e^{i\gamma/2} \lim_{\lambda\to-\infty} e^{\lambda}\Re\left(\lambda, \alpha=1\right) \tag{37}$$

In summary, what we have done is create a faithful representation of the braid group by taking the braid limit. In this way the Yang Baxter equation will be a representation of the braid equation, and for each σ_i of the braid group we associate an operator $\Re_i$:

$$\sigma_i^{\pm 1} \to \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \underbrace{\Re^{\pm \mathbb{1}}}_{(i,i+1)} \otimes \ldots \otimes \mathbb{1} \tag{38}$$

where the $+1$ indicates $\Re$ from equation (36), while the -1 its inverse given by equation (37).

6 The Yang Baxter Algebra

Finally we introduce a topic that will be used extensively next week. Basically it is the realization that the Boltzmann weights R and the monodromy matrices

T constitute and algebra, the so called “Yang Baxter Algebra”, which we will denote by Y.

In this algebra the different $T(\lambda)_i^j$ will be the generators of the algebra, while the Boltzmann weights R will play the role of structure constants. Meanwhile, the RTT=TTR relation we proved earlier, will play the role of Jacobi identity of the algebra.

This algebra is called a bi-algebra due to the fact that it has a co-product defined in the following way:

$$\Delta : Y \to Y \otimes Y$$

$$T(\lambda)_i^j \to \sum_k T(\lambda)_i^k \otimes T(\lambda)_k^j$$

This co-product leaves the Yang Baxter equation invariant.

For the case of the six vertex model we have 4 generators, which we will now denote as follows:

$$\begin{aligned} T(\lambda)_0^0 &= A(\lambda) \qquad T(\lambda)_1^0 = B(\lambda) \\ T(\lambda)_0^1 &= C(\lambda) \qquad T(\lambda)_1^1 = D(\lambda) \end{aligned}$$

As always, we can construct the adjoint representation, in which the structure constants provide a representation of the algebra. This will be given by:

$$\left(T(\lambda)_i^j\right)_l^k = R(\lambda)_{il}^{jk} \tag{39}$$

Using the R matrix for the six vertex model we find the following representations for the 4 generators of the Yang Baxter algebra:

$$A(\lambda) = \begin{bmatrix} a(\lambda) & 0 \\ 0 & b(\lambda) \end{bmatrix} \qquad B(\lambda) = \begin{bmatrix} 0 & 0 \\ c(\lambda) & 0 \end{bmatrix}$$

$$C(\lambda) = \begin{bmatrix} 0 & c(\lambda) \\ 0 & 0 \end{bmatrix} \qquad D(\lambda) = \begin{bmatrix} b(\lambda) & 0 \\ 0 & a(\lambda) \end{bmatrix}$$

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